Tsunami physics[1]

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A violent tsunami hit the coastal regions of the Indian Ocean on December 26, 2004, causing extensive damage and appalling loss of life. Many people imagine that a tsunami must be a monster wave with a very special physics, but it is in fact quite ordinary, as I shall now demonstrate. *A tsunami is just a shallow-water gravity wave with tiny amplitude and extremely large wavelength.* Such waves have a simple and well-known form, for example described in [1, ch. 24]. In the following I shall attempt to quantify the phenomenon through simple estimates based on mass and energy conservation. In view of the uncertainty in the data on the earthquake and the tsunami it generated, this is probably as much as can be done without proper simulations (see [2, 3]). Due to the great current interest, I have tried to cite as authoritative and easily accessible web-links as it has been possible for me to find.

Size of the earthquake

The epicentre of the Indonesian earthquake was situated about 160 km from the northen tip of Sumatra. Its magnitude was determined by seismographic measurements and after some discussion it landed at $\mathcal{M} = 9.0$ at the Richter scale [4, 3]. There exists an empirical relation, called the Gutenberg-Richter formula, between an earthquake's magnitude and the energy that is radiated all over the globe in the form of seismic vibrations. If the energy E is measured in joule (J), the Gutenberg-Richter formula is traditionally written as [4, 5],

$$\log_{10} E \approx 4.8 + 1.5\,\mathcal{M} \,. \tag{1}$$

For the Indonesian earthquake this formula yields $E \approx 2 \times 10^{18}$ J, corresponding to the energy released in the explosion of about 500 million tons TNT[2]. To this must be added the energy involved in the massive displacement of crustal material, but that is harder to estimate. Compared to the world energy consumption of 4×10^{20} joule per year [7], the seismically radiated energy thus corresponds to the world consumption in a little less than two days.

Great earthquakes occur less often than small. The number of earthquakes larger than a given magnitude $\mathcal{M}$ also obeys an empirical Gutenberg-Richter relation (see for example [8, fig. 5])

$$\log_{10} N \approx 8 - \mathcal{M} \,. \tag{2}$$

According to this formula, an earthquake with $\mathcal{M} \geq 9$ occurs on average only once every ten years.

Energy of a "waterberg"

The distribution of aftershocks showed that seismic displacements took place along 1200 km of the mainly north-south running fault line where the Indian plate slides under the Burmese plate [9]. The seafloor is estimated to have risen a few meters vertically upwards in a region 150 km wide, east of the subduction line, whereas the horizontal shift is estimated to be about 20 m [3].

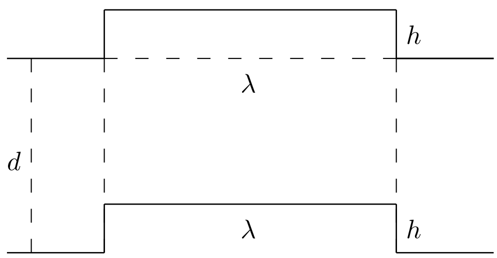

Figure 1: A box-shaped "waterberg" created by a vertical shift of height h in the seafloor over and area $\lambda \times L$ where λ is the width of the box and L its length (into the paper). The depth of the sea is d.

When the seafloor rapidly is shifted vertically, the water will be displaced and set into motion in a complicated way. But when the earthquake is of short duration, it appears to be reasonable to assume that the total mechanical energy in the water is the same, independent of how it is distributed on potential and kinetic energy. We shall therefore estimate the mechanical energy

[1] Printed in the January issue of Kvant, a Danish semipopular science journal.

[2] The energy of very large explosions (for example from nuclear weapons) is often related to the energy that is liberated in the explosion of one ton of the chemical explosive TNT (trinitrotoluene). The unit 1 ton TNT is defined to be equal to one billion calories, or 4.184×10^9 J [6].

by assuming that the whole column of water above the active area is shifted vertically upwards together with the seafloor. Assuming that the "waterberg" ends by being at rest, it will only have potential energy (see fig. 1). Since the water column originally is in buoyant equilibrium with the surrounding sea, we only need to calculate the potential energy in the part of the water column that "sticks out" above the sea surface after the displacement.

According to this argument we only need to calculate the work that must be performed in building a box-shaped waterberg by "scraping up" water from the extended surface of the sea. Denoting the height of the box by h, the width by λ, and the length by L, the mass of the box will be $M = \rho h \lambda L$ where ρ is the mass density of water. Since the particles of the waterberg on average is only lifted by $h/2$ over the sea surface, the potential energy becomes

$$E = Mg \cdot \frac{1}{2}h = \frac{1}{2}\rho g \lambda L h^2 \ , \tag{3}$$

where g is standard gravity. A box-shaped "water-valley" of the same dimensions has actually the same potential energy, because it takes the same energy to "dig it out" of the sea surface.

For the actual tsunami we take $\lambda \approx 150$ km and $L \approx 1200$ km. Historic data for similar events [10] allow us to guess that $h \approx 5$ m, a value of roughly the same size as the Russian Tsunami Laboratory's simulation of the maximal vertical displacement of the sea surface across the Indian Ocean [2]. With $\rho \approx 1000$ kg/m^3 and $g \approx 10$ m/s^2 the total energy deposited in the water becomes $E \approx 2 \times 10^{16}$ J, according to (3). This corresponds to about 1% of the seismically radiated energy of the earthquake and is in good agreement with another estimate [9].

After the extended but not very tall waterberg is created, it begins to sink back into the sea, while setting the water into motion. Its energy will eventually end up as surface waves that mainly move orthogonally to the fault line. Some of these waves later hit the coastal regions as tsunamis. The conversion of the seafloor shift into surface waves is a complicated process that depends on the character of the seafloor disturbance and the local topography (see fig. 2).

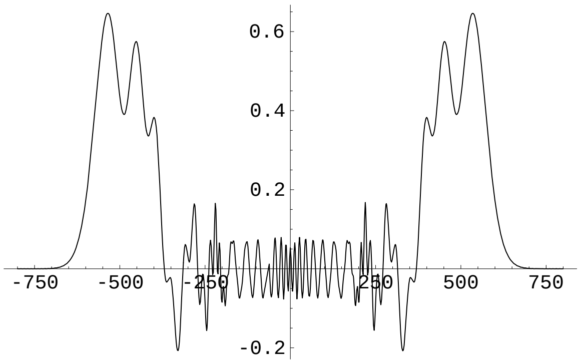

Figure 2: An attempt to simulate how a waterberg might subside and emit two wave trains moving in opposite directions. The width of the waterberg was originally $\lambda = 200$ and its height $h = 1$ in arbitrary units. The figure has been constructed by Fourier transforming the box-shaped waterberg, evolve it in time while taking into account the actual dispersion of surface waves (see below), and finally transform it back again.

Tsunamis are shallow-water waves

To continue, we shall assume that the surface waves contain wavelengths up to the width $\lambda = 150$ km of the original disturbance of the seafloor. The longest waves will thus be much longer than the average depth of the Indian Ocean, $d \approx 4$ km. It is not easy to estimate how much of the original waterberg's energy that ends up in the longest waves, but we shall in the following for definiteness choose the wave amplitude to be $a = 1.5$ m. Whatever its value, the amplitude of a tsunami wave is always much smaller than the depth of the sea, which in turn is much smaller than the wavelength. We have thus established the inequalities

$$a \ll d \ll \lambda \ , \tag{4}$$

Technically, this means that a tsunami is a *shallow-water wave with great wavelength and tiny amplitude*. The actual ratio of the wavelength to depth in the waves considered here is $\lambda/d \approx 40$, while the ratio of depth to amplitude is $d/a \approx 3000$.

Besides the wavelength λ and the amplitude a, a surface wave is also characterized by its period τ. Since in half a period, a wave crest becomes a trough, it follows that the vertical velocity must be of magnitude (see fig. 3),

$$V \sim \frac{a}{\tau} \tag{5}$$

In this and the following estimates, all purely numeric factors like 2 and π have been discarded. Therefore, these relations are not proper equations, but only comparisons of magnitudes, signalled by the sign $\sim$ (for similar).

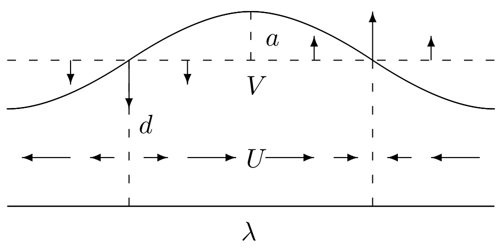

Figure 3: A single period of a surface wave moving towards the right. Notice the peculiar flow pattern that makes the wave crests (and troughs) move towards the right through a distance λ in the time τ.

In a time interval of magnitude τ the volume of the wave crest must flow down into the region below it, but since water is incompressible, it must have left the region again in the horizontal direction. Denoting the characteristic horizontal velocity by U the water will in a time τ move through the distance $U\tau$ such that the volume moving horizontally becomes $U\tau Ld$. Since this volume must be comparable with the volume coming down, of magnitude $a\lambda L$, we find

$$U\tau Ld \sim a\lambda L \ . \tag{6}$$

Solving for U we get,

$$U \sim \frac{a\lambda}{\tau d} \ . \tag{7}$$

Since $\lambda \gg d$, the horizontal flow velocity will be much larger than the vertical $V \sim a/\tau$. Notice that all the water under the wave crest moves with the characteristic velocity U, because the sea is so shallow compared to the horizontal extent of the wave. Deep-water waves with $\lambda \ll d$ on the other hand only affect the upper surface layer of thickness $\lambda/2\pi$. Generally one can estimate the properties of deep-water waves by replacing d by $\lambda/2\pi$ [1, ch. 24].

In a harmonic surface wave potential and kinetic energy are continually converted into each other. On average there must for this reason be as much potential as kinetic energy in the wave. In other words, we must for a single wave have,

$$\rho \cdot \lambda Ld \cdot U^2 \sim \rho \cdot \lambda La \cdot ga \ . \tag{8}$$

On the left hand side the kinetic energy in the water below the wave crest has been estimated as its mass times its velocity squared, and on the right hand side the potential energy of the wave crest and trough has been estimated in the same manner as for the original waterberg. All purely numeric factors have, as before, been discarded.

Inserting the velocity (7) we may solve for the period and find thereby the so-called *dispersion relation for shallow-water waves*,

$$\tau = \frac{\lambda}{\sqrt{gd}} \ . \tag{9}$$

Here the similarity sign $\sim$ has been replaced by a true equals sign, because a more careful calculation with all factors included leads to precisely this result [1, s. 344].

The speed of the wave crest, also called the *phase velocity* is for all harmonic waves given by the ratio of the wavelength to the period,

$$c = \frac{\lambda}{\tau} = \sqrt{gd} \ . \tag{10}$$

Choosing $\lambda = 150$ km and $d = 4$ km, we find the phase velocity of the tsunami, $c = 200$ m/s $=$ 720 km/h, which is comparable to the speed of a passenger jet. The period becomes $\tau =$ 750 s, about 12 minutes, and the horizontal flow velocity becomes merely $U \sim 8$ cm/s, whereas the vertical flow velocity becomes vanishingly small, of magnitude millimeters per second.

These estimates are all approximative and depend strongly on the choice of input parameters, but are nevertheless typical of the phenomenon. Satellite images seem to point to a somewhat larger wavelength ($\lambda \approx 400 - 500$ km), and smaller amplitude ($a \approx 60$ cm) [9]. Whereas the phase velocity would be the same, the period would be correspondingly longer, about 40 minutes. Furthermore, it should be noted that from being a nearly straight wave, the tsunami becomes more and more circular and longer as it proceeds over the ocean. Since the energy is fairly constant, such a stretching of the wave can only be compensated for through a diminished amplitude.

The picture that emerges is that in a tsunami wave, huge masses of water are set into motion with quite small velocity. With a wavelength in the hundreds of kilometers, the small elevation or depression of the sea surface of about a meter is completely imperceptible far at sea[3]. But even if a tsunami is quite gentle at the open sea, the colossal energy it carries turns it into a monster when it hits a coast.

Energy in a tsunami wave

The total energy in a single wavelength of a harmonic wave can be estimated as in eq. (3), and when all numeric factors are included it becomes [1, p. 351]

$$E = \frac{1}{2}\rho g \lambda L a^2 \ . \tag{11}$$

Taking $L = 1200$ km, $\lambda = 150$ km and $a = 1.5$ m, we find $E \approx 2 \times 10^{15}$ J, which is about 10% of the total energy deposited in the sea by the earthquake. Considering that there will probably be formed more than one wave in the two directions, and that a spectrum of waves with shorter wavelengths will be left behind, this seems reasonable and fits quite well with the simulation from the Tsunami Laboratory [2].

Per meter of the wave's transverse length L the energy becomes $E/L \approx 2 \times 10^9$ J/m. With such an energy at hand one would be able to lift 1.000 tons material 200 meter vertically for each meter of coastline. Maybe it becomes even clearer when one considers the energy transport in the wave. On the average it may be estimated to be $E/L\tau \approx 2\times10^6$ W/m, so that the wave continually delivers of the order of a couple of megawatts to each meter of coastline.

Dispersion

The interesting fact about shallow-water waves is that the phase velocity (10) only depends on the depth, and not on the wavelength. All shallow-water waves move in the first approximation with the same speed, and a composite wave containing many harmonic components of different wavelengths will tend to keep its shape. An ideal shallow-water wave is for this reason said to be *non-dispersive*.

But that is only an idealization. When the wavelength becomes comparable to the depth, $\lambda \sim d$, *dispersion* will set in. The relevant parameter for estimating the importance of dispersion is $2\pi d/\lambda$, which is in fact not very small. In the case discussed here with $\lambda = 150$ km and $d = 4$ km we find $2\pi d/\lambda = 0.17$. A calculation with all numeric factors included shows that a shallow-water wave with wavelength 100 km has about 0.5% smaller phase velocity than one with wavelength 150 km. During a period of 12 minutes, the crests of the shorter wave will fall behind the crests of the longer by about one kilometer. Over long distances the dispersion will separate the wavelengths such that the very longest waves hit the coast first. Provided the amplitude is the same (and that is by no means clear) the longest waves will contain most energy and cause most damage.

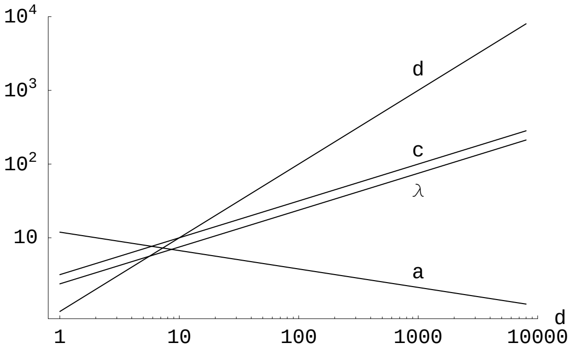

Figure 4: Variation in the wave parameters as a function of depth d, when $a = 1.5$ m, $\lambda = 150$ km and $c = 200$ m/s for $d = 4000$ m. The approximation becomes invalid when the amplitude becomes comparable to the water depth, which happens roughly at the place where the line "d" crosses the line "a". Notice that the different quantities in the figure are measured in different units (as indicated above).

When the tsunami reaches land

Imagine that a shallow-water wave with an infinite sequence of wave crests is on its way towards a coastline. Assuming that the depth d of the sea slowly diminishes towards land, the phase velocity will decrease as $c \sim \sqrt{d}$ (disregarding

[3]Today the high precision of modern navigation instruments (GPS etc) should allow larger ships to detect the small change in surface height as well as the slow drift due the horizontal flow.

thereby the influence of a sharp continental shelf). Since there is no place that wave crests can accumulate, the period τ between the wave crests must be the same all the way in. According to the dispersion relation (9), this means that the wavelength becomes shorter in the same way as the phase velocity $\lambda \sim \sqrt{d}$. Every wavelength contains the energy(11), and since long waves dissipate very little energy, it follows that the energy $E \sim \lambda a^2$ of a wavelength (or the rate of energy transport E/τ) must be constant all the way in, provided reflection from the sloping bottom can be ignored. The amplitude must accordingly grow as $a \sim 1/\sqrt{\lambda} \sim d^{-1/4}$, a relation called Green's law.

Let us as before consider a tsunami with a period of 12 minutes, which for $d = 4000$ m has wavelength $\lambda = 150$ km, phase velocity $c = 200$ m/s and amplitude $a = 1.5$ m. When it arrives at the depth $d = 40$ m which is a factor 100 smaller than in the ocean at large, its wavelength will have decreased by a factor 10 to $\lambda = 15$ km, and similarly for the phase velocity $c = 20$ m/s $= 72$ km/h, whereas the amplitude has risen to $a = 5$ m (se fig. 4). Closer to land, the wave rises still more and moves even slower. When the amplitude becomes comparable to the depth, non-linear effects set in and may cause the wave to break and froth before it hits the beach. Here its enormous energy of about one gigajoule per meter of beach causes extensive destruction of the coastal areas.

Conclusion

Tsunami waves are shallow-water waves with very large wavelengths, phase velocities and periods. Whereas usual ocean swells may have wavelengths up to 150 meter and last a few seconds, tsunamis at oceanic depths may have wavelengths in the hundreds of kilometers, periods from 10 minutes up to an hour, and phase velocities comparable to the speed of passenger jets.

When such a wave reaches land, its period will remain the same as in deep waters, whereas the phase velocity and the wavelength become smaller while the amplitude grows. The net result is that the water level is raised several meters for half a period, flooding the beach and coastal areas. If the tsunami contains more than one wave, the troughs separating them will cause a strong surge of the flood waters towards the sea.

The behavior of a tsunami wave is strongly dependent on local topography. In the Indonesian earthquake, the fault line went roughly from south to north, and the tsunami waves started out towards the east and west. This explains probably, why Bangladesh escaped almost unhurt from the event, even if flooding often plagues this low-lying country.

Depending on the character of the seismic disturbance of the seafloor, the tsunami wave train may be preceded by a trough which lowers the water level on the beach before the arrival of the crest. In the actual case, this phenomenon was mainly seen east of the fault line, for example in Thailand, and on the west coast of Sri Lanka [11]. People on the beach only heeded this warning in a few locations.

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