Answers to extra problems

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1 Continuous matter

1.5 **(a)** Use that

$$\int_{-\infty}^{\infty} e^{-\beta v^2} dv = \sqrt{\pi/\beta}.$$

(b) Use that $dv_x dv_y dv_z = 4\pi v^2\, dv$ and

$$\int_0^\infty v^2 e^{-\beta v^2}\, dv = -\frac{d}{d\beta}\int_0^\infty e^{-\beta v^2}\, dv = -\frac{d\left(\frac{1}{2}\sqrt{\pi/\beta}\right)}{d\beta} = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}}$$

(c)

$$\langle v\rangle \equiv \int_0^\infty v p(v)\, dv = 4\pi\left(\frac{\beta}{\pi}\right)^{3/2}\int_0^\infty v^3 e^{-\beta v^2}\, dv = 4\pi\left(\frac{\beta}{\pi}\right)^{3/2}\frac{1}{2\beta^2} = \frac{2}{\sqrt{\pi\beta}},$$

$$\langle v^2\rangle \equiv \int_0^\infty v^2 p(v)\, dv = 4\pi\left(\frac{\beta}{\pi}\right)^{3/2}\int_0^\infty v^4 e^{-\beta v^2}\, dv = 4\pi\left(\frac{\beta}{\pi}\right)^{3/2}\frac{d^2}{d\beta^2}\int_0^\infty e^{-\beta v^2}\, dv$$

$$= 4\pi\left(\frac{\beta}{\pi}\right)^{3/2}\frac{d^2\left(\frac{1}{2}\sqrt{\pi/\beta}\right)}{d\beta^2} = \frac{3}{2\beta}.$$

(d) Use that $\sigma^2 = \langle v^2\rangle - \langle v\rangle^2$.
(e) Using that $M_{\text{air}} = mN_A = 29\ \text{g mol}^{-1}$ and $R_{\text{mol}} = kN_A = 8.3\ \text{J K}^{-1}\,\text{mol}^{-1}$ one finds for $T = 300$ K and $p = 1$ atm the values:

$$\langle v\rangle = 468\ \text{m s}^{-1}, \qquad \sqrt{\langle v^2\rangle} = 508\ \text{m s}^{-1}, \qquad \sigma = 197\ \text{m s}^{-1}$$

25 Gravity waves

25.8 **(a)** Expanding the finite-depth velocity potential in Equation CM-(25.29) to leading non-trivial order in $kd \ll 1$, one gets the expansion

$$\Psi \approx ac\frac{1+\frac{1}{2}(kd\zeta)^2}{kd+\frac{1}{6}(kd)^3}\sin(kx-\omega t) \approx \frac{ac}{kd}\left(1+\tfrac{1}{6}(kd)^2(3\zeta^2-1)+\mathcal{O}\left((kd)^4\right)\right)\sin(kx-\omega t),$$

where $\zeta = 1 + z/d$. From this we get

$$v_x = \frac{\partial\Psi}{\partial x} = \frac{ac}{d}\left(1+\mathcal{O}\left((kd)^2\right)\right)\cos(kx-\omega t),$$

$$v_z = \frac{\partial\Psi}{\partial z} = kac\left(\zeta+\mathcal{O}\left((kd)^2\right)\right)\sin(kx-\omega t).$$

(b) The divergence becomes

$$\nabla_x v_x + \nabla_z v_z = \frac{ac}{d^2}\mathcal{O}\left((kd)^3\right). \tag{1.1}$$

The right hand side is very small for $kd \ll 1$. Successive approximations will make the right-hand side smaller.