Channel entrance flow

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A two-dimensional semi-infinite channel between parallel plates a distance $2a$ apart carries an incompressible steady flow with average velocity U. The entrance length L has been estimated on page PCM-265 to be of the form,

$$\frac{L}{2a} \approx k\,\mathsf{Re}, \qquad\qquad \mathsf{Re} = \frac{2aU}{\nu}, \tag{1}$$

with $k \approx 1/16 \approx 0.063$ for $\mathsf{Re} \gg 1$. The goal of the following calculation is to obtain a better estimate by emulating the calculation of Langhaar for the case of pipe entrance flow[1].

We first establish the geometry and the exact Navier-Stokes equations, as well as the symmetries and the boundary conditions. Next we show that Prandtl's boundary layer equations (page PCM-486) are valid with small modifications. Integral relations are derived, and a velocity profile is derived, leading to the final result $k = 0.039$.

Simulated channel entrance flow at $\mathsf{Re} = 100$. The fluid enters on top and exits at the bottom. Note how the streamlines (light) converge towards the center while the pressure contours (dark) flatten out as the Poiseuille profile is established downstream.

1 Navier-Stokes equations in 2D

In the region $(0 \le x < \infty, -a \le y \le a)$, channel flow satisfies the continuity equation,

$$\nabla_x v_x + \nabla_y v_y = 0, \tag{2}$$

and the Navier-Stokes equations,

$$(v_x\nabla_x + v_y\nabla_y)v_x = -\nabla_x p/\rho + \nu(\nabla_x^2 + \nabla_y^2)v_x, \tag{3}$$

$$(v_x\nabla_x + v_y\nabla_y)v_y = -\nabla_y p/\rho + \nu(\nabla_x^2 + \nabla_y^2)v_y, \tag{4}$$

where ρ is the (constant) density. Symmetry demands that v_x is even in y and v_y is odd, whereas the pressure is even,

$$v_x(x,-y) = v_x(x,y), \qquad v_y(x,-y) = -v_y(x,y), \qquad p(x,-y) = p(x,y). \tag{5}$$

This limits the region of interest to $(0 \le x < \infty, 0 \le y \le a)$.

The boundary conditions become

$$v_x = U, \qquad v_y = 0, \qquad (x = 0, 0 \le y \le a), \tag{6}$$

$$v_x = \tfrac{3}{2}U(1 - y^2/a^2), \qquad v_y = 0, \qquad (x = \infty, 0 \le y \le a), \tag{7}$$

$$\nabla_y v_x = 0, \qquad v_y = 0, \qquad (0 \le x \le \infty, y = 0), \tag{8}$$

$$v_x = 0, \qquad v_y = 0, \qquad (0 \le x \le \infty, y = a). \tag{9}$$

The boundary values for the pressure are determined by these (up to a constant).

[1] H. L. Langhaar, *Steady flow in the transition length of a straight tube*, Journal of Applied Mechanics **64** (1942) A55–A58.

Without loss of generality we shall in the following choose units such that

$$\rho = U = a = 1 \tag{10}$$

For $\mathsf{Re} \gg 1$ eq. (1) implies that $L \sim 1/\nu \sim \mathsf{Re}$.

2 The Prandtl approximation

In making estimates we note that $v_x \sim 1$ in the bulk of the flow. Using that $\nabla_x \sim 1/L \sim \nu$ and $\nabla_y \sim 1$, the continuity equation implies that in the bulk of the flow,

$$v_y \sim \nu. \tag{11}$$

Assuming a long entrance region, $L \gg 1$, the transverse velocity is always small.

Multiplying the first NS-equation by L leads to the following estimate of the x-variation in pressure from advection and viscosity

$$\Delta_x p \sim 1 + \nu L \sim 1. \tag{12}$$

We have dropped the double x-derivative in the Laplacian, because it is of order $1/L^2 \sim \nu^2$ relative to the double y-derivative. In dimensionfull variables we have $\Delta_x p \sim \rho U^2$.

The second NS equation then leads to a similar estimate of the transverse variation in pressure,

$$\Delta_y p \sim \frac{1}{L^2} + \frac{\nu}{L} \sim \nu^2. \tag{13}$$

This reflects the usual stiffness of the pressure in a boundary layer, so that we may assume that the pressure only depends on x, or $p = p(x)$ up to errors of relative order ν^2.

The first NS-equation now becomes the Prandtl equation (with relative errors of order ν^2)

$$\boxed{v_x \nabla_x v_x + v_y \nabla_y v_x = G + \nu \nabla_y^2 v_x,} \tag{14}$$

where the pressure gradient, $G(x) = -p'(x)/\rho$, only depends on x.

For isolated boundary layers one uses Bernoulli's theorem to identify $G(x)$ with $u(x)u'(x)$ where $u(x)$ is the slip-flow velocity, but that is not possible here because the boundary layers from the two sides merge at the center of the channel with a so far unknown central velocity, $u(x) = v_x(x, 0)$. Setting $y = 0$ in the Prandtl equation above, we get

$$\boxed{G = u \nabla_x u - \nu \left[\nabla_y^2 v_x\right]_{y=0}.} \tag{15}$$

Since curvature of the central velocity profile is always non-zero, the second term is always non-vanishing, so the conventional slip-flow expression is clearly invalid here.

Global mass conservation

From the continuity equation we get after integrating with respect to y,

$$v_y = -\nabla_x \int_0^y v_x(x, y')\, dy'. \tag{16}$$

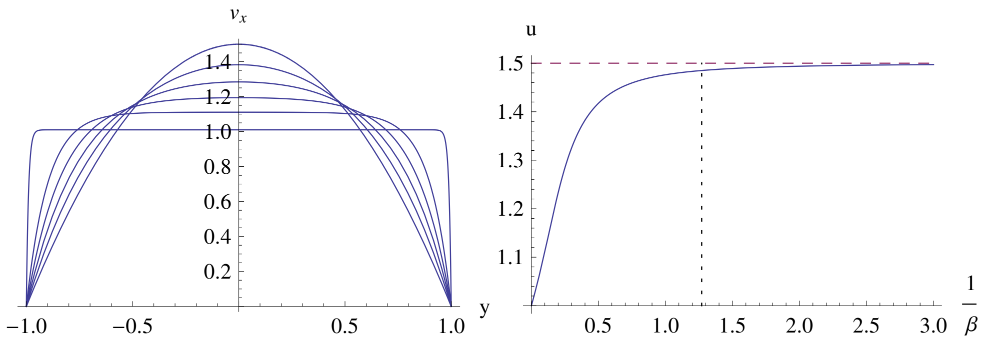

Figure 1. Left: Velocity profiles as functions of y for $\beta = 10^{-6}, 2.5, 4, 6, 10, 100$. One notices the lack of a small dip in the middle of each curve, as shown by simulations (page PCM-265). The approximation is the least trustworthy close to the entrance. **Right:** Central flow velocity as a function of $1/\beta$. It reaches 99% of the terminal value (3/2) for $\beta = \beta_{99} = 0.787076$, or $1/\beta_{99} = 1.27052$ (dotted line).

Since $v_y = 0$ at $y = 1$ it follows that $\int_0^1 v_x(x, y)\, dy$ is independent of x. Its value can trivially be calculated for $x = 0$ where $v_x = 1$, so that

$$\boxed{\int_0^1 v_x(x, y)\, dy = 1,} \tag{17}$$

It expresses mass conservation, or equivalently that the average velocity is always 1.

Global momentum balance

Combining the Prandtl equation and the continuity equation we get

$$\nabla_x(v_x^2) + \nabla_y(v_x v_y) = G + \nu \nabla_y^2 v_x, \tag{18}$$

and integrating both sides over the interval $0 < y < 1$ we find, again using the boundary conditions,

$$\boxed{\nabla_x \int_0^1 v_x^2 dy = G + \nu \left[\nabla_y v_x\right]_{y=1},} \tag{19}$$

which expresses exact global momentum balance.

3 Approximative theory

It is not possible to solve the Prandtl equations as they stand, but we shall now obtain an approximative solution. Such a solution will a priori not satisfy the conditions of global mass conservation and momentum balance. But if these conditions are broken, the solution is physically worthless. So mass conservation and momentum balance must be imposed on approximative solution to secure the internal consistency.

To obtain an approximation to the solution we first define the function

$$H = v_x \nabla_x v_x + v_y \nabla_y v_x - \nu\beta^2 v_x, \tag{20}$$

where $\beta = \beta(x)$ is an unknown function of x. We shall with Langhaar assume that H is essentially independent of y, that is, $\nabla_y H \approx 0$. After finding the solution, we shall investigate to which extent this condition is fulfilled[2].

With this notation Prandtl's equation (14) may be written

$$\nu \nabla_y^2 v_x = \nu\beta^2 v_x + H - G. \tag{21}$$

For $H = H(x)$ the solution, which must be symmetric under $y \to -y$, becomes

$$v_x = \frac{G - H}{\nu\beta^2} + C \cosh \beta y \tag{22}$$

where C is an arbitrary function of x. It is determined by the boundary condition $v_x = 0$ for $y = 1$, and we find

$$v_x = \frac{G - H}{\nu\beta^2}\left(1 - \frac{\cosh \beta y}{\cosh \beta}\right). \tag{23}$$

Imposing mass conservation (17), we get

$$G - H = \frac{\nu\beta^2}{1 - \tanh \beta/\beta}, \tag{24}$$

so that

$$\boxed{v_x = \frac{1 - \cosh \beta y / \cosh \beta}{1 - \tanh \beta/\beta}.} \tag{25}$$

One may verify that for $0 < y < 1$ this expression has the correct boundary value $v_x \to 1$ for $\beta \to \infty$ (that is for $x \to 0$), and $v_x \to \frac{3}{2}(1 - y^2)$ for $\beta \to 0$ (that is for $x \to \infty$). The velocity profiles are shown in Figure 1.

The central velocity is now obtained by setting $y = 0$,

$$\boxed{u = \frac{1 - \mathrm{sech}\beta}{1 - \tanh \beta/\beta}} \tag{26}$$

where sech $= 1/\cosh$ is the hyperbolic secans.

Relation between x and β

What remains is to determine β as a function of x from momentum balance (19). In practice, it is easier to determine $x = x(\beta)$ and then find the inverse.

Eliminating the pressure gradient G by means of (15), momentum balance becomes

$$\nabla_x \left(\langle v_x^2 \rangle - \tfrac{1}{2}u^2\right) = \nu \left[\nabla_y v_x\right]_{y=1} - \nu \left[\nabla_y^2 v_x\right]_{y=0} \tag{27}$$

where

$$\langle v_x^2 \rangle = \int_0^1 v_x^2 dy \tag{28}$$

is the average squared velocity along x.

[2]Various arguments can be given. It is fulfilled for the limiting flow for $x \to \infty$, which has $v_x \to \frac{3}{2}(1 - y^2)$ and $v_y \to 0$. Provided $\beta(x) \to 0$ for $x \to \infty$, we find $H \to 0$ in this limit. It is also fulfilled everywhere in the central region as yet untouched by the growing boundary layers, because there v_x is flat and depends mainly on x, while $v_y \approx 0$. Langhaar actually assumed that $H = 0$, but that is, as we shall see, not fulfilled by the solution.

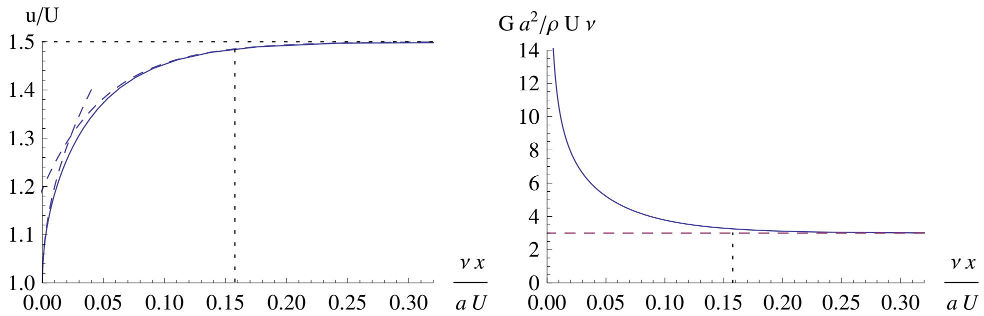

Figure 2. Central velocity (left) and central gradient (right). The dashed lines are the asymptotic approximations. The vertical dotted line is the 99% asymptotic point. Notice that the axes have been dimensionalized.

Differentiating with respect to x through β we obtain

$$\nu \frac{dx}{d\beta} = -h(\beta), \qquad h = \frac{f'(\beta)}{g(\beta)} \tag{29}$$

where

$$f \equiv \langle v_x^2 \rangle - \tfrac{1}{2}u^2 = \frac{3/2}{1 - \tanh\beta/\beta} + \frac{\operatorname{sech}\beta - 1}{(1 - \tanh\beta/\beta)^2} \tag{30}$$

$$g \equiv \left[\nabla_y^2 v_x\right]_{y=0} - \left[\nabla_y v_x\right]_{y=1} = \beta^2 \left(\frac{1 - \operatorname{sech}\beta}{1 - \tanh\beta/\beta} - 1 \right) \tag{31}$$

It is fairly straightforward to evaluate $f'(\beta)$.

Using that $x \to 0$ for $\beta \to \infty$ we finally obtain the desired function,

$$\boxed{\nu x = \int_\beta^\infty h(s)\, ds,} \tag{32}$$

The integral can only be done numerically. The central velocity and pressure gradients are plotted in Figure 2.

Entry length

The leading approximations are

$$h = \begin{cases} \dfrac{73}{700}\dfrac{1}{\beta} + \mathcal{O}(\beta) & (\beta \to 0), \\ \dfrac{1}{2\beta^3} + \mathcal{O}\left(\beta^{-4}\right) & (\beta \to \infty). \end{cases} \tag{33}$$

For $\beta \to 0$ the asymptotic form is

$$\nu x = -\frac{73}{700}\log\beta + C + \mathcal{O}\left(\beta^2\right) \qquad (\beta \to 0) \tag{34}$$

where the constant C is determined by numeric integration.

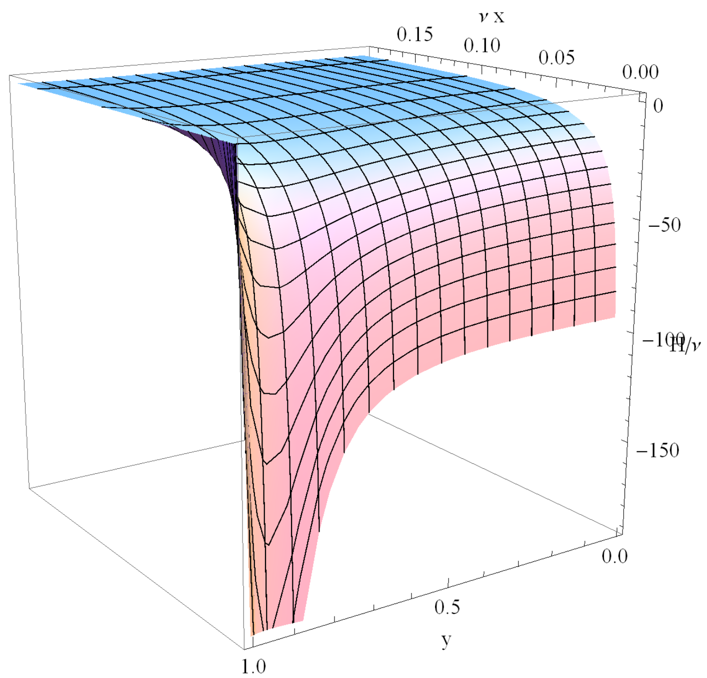

Figure 3. Error function H/ν as a function of y and νx in dimensionless units. Apart from the region near the entry $x \approx 0$ and close to the plates for $0.7 < y < 1$, the error function is nearly constant, as assumed in Equation (20).

The constant C may be expressed as a perfectly convergent integral

$$C = \lim_{\beta \to 0} \left(\frac{73}{700} \log \beta + \int_\beta^\infty h(s)\, ds \right) = \int_0^1 \left(h(s) - \frac{73}{700} \frac{1}{s} \right) ds + \int_1^\infty h(s)\, ds$$
$$= 0.130515. \tag{35}$$

Its value is very close to $\frac{3}{23} = 0.130435$.

The entry length is defined as the point where the central velocity has reached 99% of its terminal value. In the caption of Figure 1R it is determined to be $\beta_{99} = 0.78707$, corresponding to $\nu x_{99} = 0.157588$. The constant in Equation (1) then becomes $k_{99} = \nu x_{99}/4 = 0.0393971$.

If we use the asymptotic approximations for $\beta \to 0$

$$u \approx \frac{3}{2}\left(1 - \frac{\beta^2}{60}\right), \qquad \nu x \approx -\frac{73}{700} \log \beta + \frac{3}{23}, \tag{36}$$

we find instead $\beta_{99} \approx \sqrt{0.60} = 0.774597$ and $\nu x_{99} = 0.157071$. These values differ from the numeric values by less than 1%.

Error analysis

Having obtained an approximate solution, we can calculate the value of H from (20). It is plotted as a function of y and νx in Figure 3. The assumption that H is nearly independent of y is certainly confirmed, except very close to the rim and the entrance to the pipe. The calculation of the entrance length nevertheless holds, because it it carried out for $y = 0$. Langhaar's assumption that $H = 0$ is not fulfilled close to the entry.