Pipe entrance flow

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A semi-infinite pipe of radius a carries an incompressible steady flow with average velocity U. The entrance length L has been estimated on page PCM-272[1] to be of the form

$$\frac{L}{2a} \approx k\,\mathsf{Re}, \qquad \mathsf{Re} = \frac{2aU}{\nu}, \tag{1}$$

with $k \approx 1/16 \approx 0.063$ for $\mathsf{Re} \gg 1$. The goal of the following calculation is to obtain a better estimate by following Langhaar's original calculation[2] with some improvements.

We first establish the geometry and the exact Navier-Stokes equations, as well as the symmetries and the boundary conditions. Next we derive Prandtl's boundary layer equations for this symmetric three-dimensional system (page PCM-486). Integral relations are derived, and a certain velocity profile is found, leading to the final result $k = 0.057$.

1 Navier-Stokes equations

Rotationally invariant pipe flow is defined in the region $0 \le z < \infty$ times $0 \le r \le a$, and satisfies the continuity equation as well as the radial and longitudinal Navier-Stokes equations,

$$\nabla_z v_z + \frac{1}{r}\nabla_r(r v_r) = 0 \tag{2}$$

$$v_z \nabla_z v_z + v_r \nabla_r v_z = -\nabla_z p/\rho + \frac{\nu}{r}\nabla_r\left(r\nabla_r v_z\right) + \nu\nabla_z^2 v_z \tag{3}$$

$$v_z \nabla_z v_r + v_r \nabla_r v_r = -\nabla_r p/\rho + \nu\nabla_r\left(\frac{1}{r}\nabla_r(r v_r)\right) + \nu\nabla_z^2 v_r \tag{4}$$

where ρ is the constant density.

Boundary conditions are

$$v_z = U, \qquad v_r = 0, \qquad (z = 0, 0 \le r \le a) \tag{5}$$

$$v_z = 2U(1 - r^2/a^2), \qquad v_r = 0, \qquad (z = \infty, 0 \le r \le a) \tag{6}$$

$$\nabla_r v_z = 0, \qquad v_r = 0, \qquad (0 \le z \le \infty, r = 0) \tag{7}$$

$$v_z = 0, \qquad v_r = 0, \qquad (0 \le z \le \infty, r = a) \tag{8}$$

Apart from a constant, the boundary conditions on the pressure are determined by these.

Without loss of generality we shall in the following choose units such that

$$\rho = U = a = 1 \tag{9}$$

For $\mathsf{Re} \gg 1$ eq. (1) implies that $L \sim 1/\nu \sim \mathsf{Re}$ up to a numeric factor of order unity.

[1] The prefix PCM refers to B. Lautrup, Physics of Continuous Matter, Second Edition, Taylor&Francis (2011).

[2] H. L. Langhaar, *Steady flow in the transition length of a straight tube*, Journal of Applied Mechanics **64** (1942) A55–A58.

2 The Prandtl approximation

In making estimates we note that $v_z \sim 1$ in the bulk of the flow. Using that $\nabla_z \sim 1/L \sim \nu$ and $\nabla_r \sim 1$, the continuity equation implies that in the bulk of the flow,

$$v_r \sim \nu. \tag{10}$$

Assuming a long entrance region, $L \gg 1$, the radial velocity is always small.

Multiplying the first NS-equation by L leads to the following estimate of the z-variation in pressure from advection and viscosity

$$\Delta_z p \sim 1 + \nu L \sim 1. \tag{11}$$

We have dropped the double z-derivative in the Laplacian, because it is of order $1/L^2 \sim \nu^2$ relative to the double r-derivative. In dimensionfull variables we have $\Delta_z p \sim \rho U^2$.

The second NS equation then leads to a similar estimate of the transverse variation in pressure,

$$\Delta_r p \sim \frac{1}{L^2} + \frac{\nu}{L} \sim \nu^2. \tag{12}$$

This reflects the usual stiffness of the pressure in a boundary layer, so that we may assume that the pressure only depends on z, or $p = p(z)$, up to errors of relative order ν^2.

The first NS-equation now becomes the Prandtl equation (with relative errors of order ν^2)

$$\boxed{v_z \nabla_z v_z + v_r \nabla_r v_z = G + \frac{\nu}{r} \nabla_r \left(r \nabla_r v_z \right),} \tag{13}$$

where the pressure gradient, $G(z) = -p'(z)/\rho$, only depends on z.

For isolated boundary layers one uses Bernoulli's theorem to identify $G(z)$ with $u(z)u'(z)$ where $u(z)$ is the slip-flow velocity, but that is not possible here because the boundary layers from the rim of the pipe merge at the center of the channel with a so far unknown central velocity, $u(z) = v_z(z, 0)$. Setting $r = 0$ in the Prandtl equation above, we get

$$\boxed{G = u \nabla_z u - \nu \left[\frac{1}{r} \nabla_r \left(r \nabla_r v_z \right) \right]_{r=0} .} \tag{14}$$

Since curvature of the central velocity profile is always non-zero, the second term is always non-vanishing, so the conventional slip-flow expression is clearly invalid here.

Global mass conservation

From the continuity equation we get after integrating with respect to r,

$$v_r = -\nabla_z \frac{1}{r} \int_0^r v_z(z, s)\, s ds. \tag{15}$$

Since $v_r = 0$ at $r = 1$ it follows that $\int_0^1 v_r(z, r)\, r dr$ is independent of z. Its value can trivially be calculated for $z = 0$ where $v_z = 1$, so that

$$\boxed{\int_0^1 v_x(z, r)\, 2 r dr = 1,} \tag{16}$$

It clearly expresses mass conservation. Defining the total discharge $Q = \int_0^1 v_z 2\pi r dr$, it also expresses that the average velocity is always $U \equiv Q/\pi a^2 = 1$).

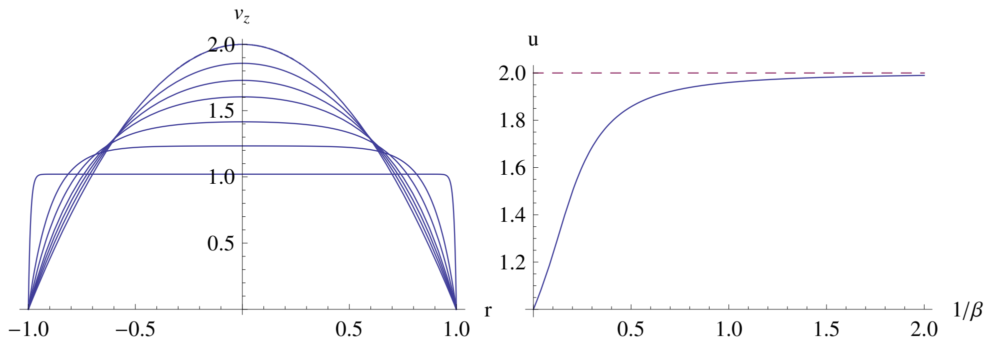

Figure 1. Left: : Velocity profiles as functions of r for $\beta = 10^{-6}, 2, 3, 4, 6, 10, 100$. One notices the lack of a small dip in the middle of each curve, as shown by simulations (page PCM-265). The approximation is least trustworthy close to the entrance. **Right:** The central flow as a function of $1/\beta$. It reaches 99% of the terminal value (2) for $\beta = \beta_{99} = 0.699861$, or $1/\beta_{99} = 1.42885$.

Global momentum balance

Combining the Prandtl equation and the continuity equation we get

$$\nabla_z(v_z^2) + \frac{1}{r}\nabla_r(r v_z v_r) = G + \frac{\nu}{r}\nabla_r\left(r\nabla_r v_z\right). \tag{17}$$

Multiplying this by $2r$ and integrating both sides over the interval $0 < r < 1$ we find, again using the boundary conditions,

$$\boxed{\nabla_z \int_0^1 v_z^2 2r dr = G + 2\nu\left[\nabla_r v_z\right]_{r=1},} \tag{18}$$

which expresses exact global momentum balance.

3 Approximative theory

It is not possible to solve the Prandtl equations as they stand, but we shall now obtain an approximative solution. Such a solution will a priori not satisfy the conditions of global mass conservation and momentum balance. But if these conditions are broken, the solution is physically worthless. So mass conservation and momentum balance must be imposed on approximative solution to secure the internal consistency.

To obtain an approximation to the solution we first define the function

$$H = v_z\nabla_z v_z + v_r\nabla_r v_z - \nu\beta^2 v_z, \tag{19}$$

where $\beta = \beta(z)$ is an unknown function of z. We shall with Langhaar assume that H is essentially independent of r, that is, $\nabla_r H \approx 0$. After finding the solution, we shall investigate to which extent this condition is fulfilled[3].

[3]Various arguments can be given. It is fulfilled for the limiting flow for $z \to \infty$, which has $v_z \to 2(1-r^2)$ and $v_r \to 0$. Provided $\beta(z) \to 0$ for $z \to \infty$, we find $H \to 0$ in this limit. It is also fulfilled everywhere in the central region as yet untouched by the growing boundary layers, because there v_z is flat and depends mainly on z, while $v_r \approx 0$. Langhaar actually assumed that $H = 0$, but that is, as we shall see, not fulfilled by the solution.

With this notation Prandtl's equation (13) may be written

$$\nu \left(\nabla_r^2 v_z + \frac{1}{r} \nabla_r v_z \right) = \nu\beta^2 v_z + H - G. \tag{20}$$

Provided H only depends on z, the only solution that is regular for $r \to 0$ is

$$v_z = \frac{G - H}{\nu\beta^2} + C I_0(\beta r) \tag{21}$$

where I_n denotes the hyperbolic (or modified) Bessel-function of n'th order, and C is an arbitrary function of z. C is determined by the boundary condition $v_z = 0$ for $r = 1$, and we find

$$v_z = \frac{G - H}{\nu\beta^2} \left(1 - \frac{I_0(\beta r)}{I_0(\beta)} \right). \tag{22}$$

Imposing mass conservation (16), we get from the standard Bessel relations

$$G - H = \nu\beta^2 \frac{I_0(\beta)}{I_2(\beta)}, \tag{23}$$

so that

$$\boxed{v_z = \frac{I_0(\beta) - I_0(\beta r)}{I_2(\beta)}.} \tag{24}$$

One may verify that for $0 < y < 1$ this expression has the correct boundary value $v_x \to 1$ for $\beta \to \infty$ (that is for $z \to 0$), and $v_z \to 2(1 - r^2)$ for $\beta \to 0$ (that is for $z \to \infty$). The velocity profiles are shown in Figure 1L.

The central velocity is now obtained by setting $r = 0$, using that $I_0(0) = 1$,

$$\boxed{u = \frac{I_0(\beta) - 1}{I_2(\beta)}} \tag{25}$$

which is shown in Figure 1R.

Relation between z and β

What remains is to determine β as a function of z from momentum balance (18). In practice, it is easier to determine $z = z(\beta)$ and then find the inverse.

Eliminating the pressure gradient G by means of (14), momentum balance becomes

$$\nabla_z \left(\left\langle v_z^2 \right\rangle - \tfrac{1}{2} u^2 \right) = 2\nu \left[\nabla_r v_z \right]_{r=1} - \nu \left[\frac{1}{r} \nabla_r \left(r \nabla_r v_z \right) \right]_{r=0} \tag{26}$$

where

$$\left\langle v_z^2 \right\rangle = \int_0^1 v_z^2 \, 2r dr \tag{27}$$

is the average squared velocity along z.

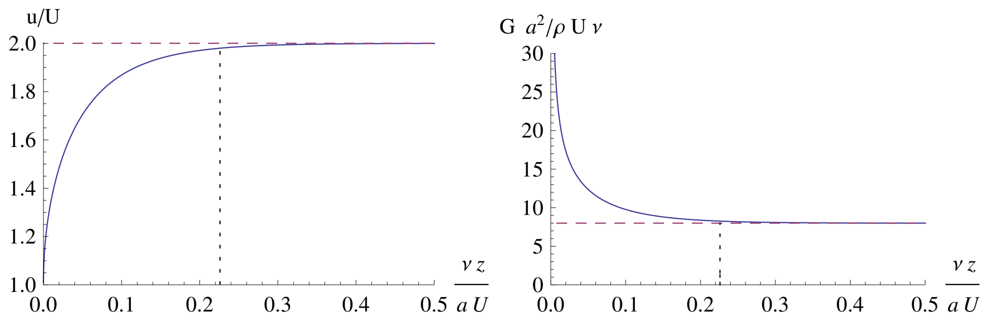

Figure 2. Central velocity u and pressure gradient G as a functions of z. The dotted line indicates the 99% point. Dimensional units have been reintroduced.

Differentiating with respect to z through β we obtain

$$\boxed{\nu \frac{dz}{d\beta} = -h(\beta), \qquad h = \frac{f'(\beta)}{g(\beta)}} \tag{28}$$

where we have taken into account that z is always a decreasing function of β. The numerator and denominator functions are:

$$f \equiv \langle v_z^2 \rangle - \tfrac{1}{2}u^2 = \frac{4I_0 I_2 - (I_0 - 1)^2 - 2I_1^2}{2I_2^2} \tag{29}$$

$$g \equiv \left[\frac{1}{r}\nabla_r \left(r \nabla_r v_z\right)\right]_{r=0} - 2\left[\nabla_r v_z\right]_{r=1} = \beta^2 \frac{I_0 - 1 - I_2}{I_2} \tag{30}$$

where I_1 is the Bessel function of 1'st order.

The solution is

$$\boxed{\nu z = \int_\beta^\infty h(s)\, ds,} \tag{31}$$

Using this, the central velocity u and central gradient G (from (14)) are plotted in Figure 2 as functions of z.

Entry length

In the limits the integrand becomes,

$$h = \begin{cases} \dfrac{5}{36\beta} + \mathcal{O}(\beta) & \beta \to 0, \\ \dfrac{1}{2\beta^3} + \mathcal{O}\left(\beta^{-4}\right) & \beta \to \infty. \end{cases} \tag{32}$$

Clearly the integral has for $\beta \to 0$ the asymptotic form

$$\nu z = -\frac{5}{36}\log\beta + C + \mathcal{O}\left(\beta^2\right), \tag{33}$$

where C is a constant.

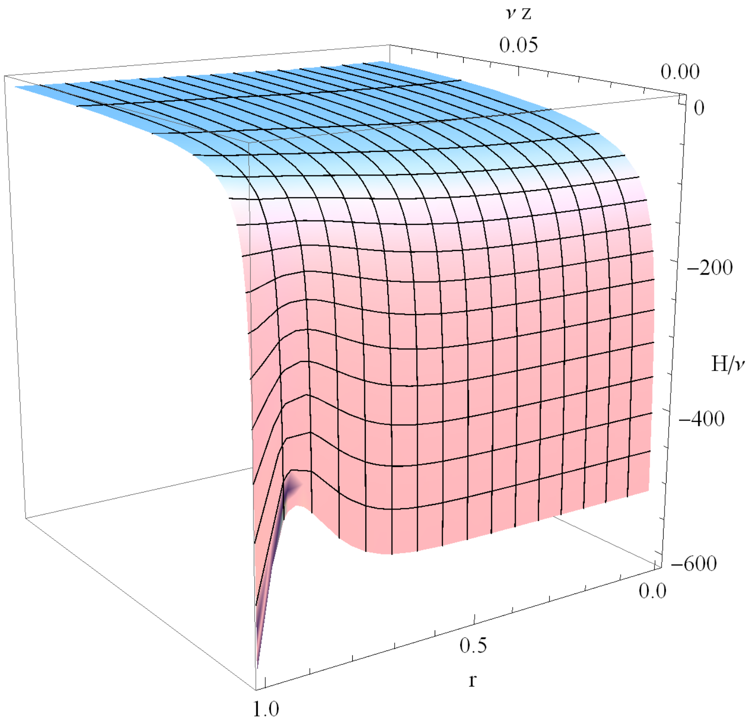

Figure 3. Error function H/ν as a function of r and νz in dimensionless units. Apart from the region near the entry $z \approx 0$, close to the rim for $0.7 < r < 1$, the error function is nearly constant, as assumed in Equation (19).

The constant C may be expressed as a perfectly convergent integral

$$C = \lim_{\beta \to 0} \left(\frac{5}{36} \log \beta + \int_\beta^\infty h(s)\, ds \right) = \int_0^1 \left(h(s) - \frac{5}{36s} \right) ds + \int_1^\infty h(s)\, ds$$
$$= 0.174925. \tag{34}$$

Its value is very close to $\frac{7}{40} = 0.175$.

The entry length is defined as the point where the central velocity has reached 99% of its terminal value. In Figure 1R it is determined to be $\beta_{99} = 0.699861$, corresponding to $\nu z_{99} = 0.226613$. The constant in Equation (1) then becomes $k_{99} = \nu z_{99}/4 = 0.0569939$.

If we use the asymptotic approximations for $\beta \to 0$

$$u \approx 2\left(1 - \frac{\beta^2}{48}\right), \qquad \nu z \approx -\frac{5}{36} \log \beta + \frac{7}{40}, \tag{35}$$

we find instead $\beta_{99} \approx \sqrt{0.48} = 0.69282$ and $\nu z_{99} = 0.22597$. These values differ from the exact numeric values by less than 1%.

Error analysis

Having obtained an approximate solution, we can calculate the value of H from (19). It is plotted as a function of r and β in Figure 3. The assumption that H is nearly independent of r is certainly confirmed, except very close to the rim and the entrance to the pipe. Langhaar's assumption that $H = 0$ is not fulfilled, except far downstream that Poisseuille flow has set in. The calculation of the entrance length nevertheless holds, because it it carried out for $r = 0$.