

1

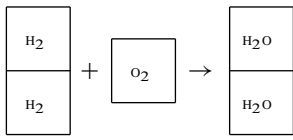
Continuous matter

The everyday experience of the smoothness of matter is an illusion. Since the beginning of the twentieth century it has been known with certainty that the material world is composed of microscopic atoms and molecules, responsible for the macroscopic properties of ordinary matter. Long before the actual discovery of molecules, chemists had inferred that something like molecules had to exist, even if they did not know how big they were. Molecules *are* small—so small that their existence may be safely disregarded in all our daily doings. Although everybody possessing a powerful microscope will note the irregular Brownian motion of small particles in a liquid, it took quite some mental effort to move from the everyday manipulation of objects and recognize that this is a sign that molecules are really there.

Continuum physics deals with the systematic description of matter at length scales that are large compared to the molecular scale. Most macroscopic length scales occurring in practice are actually huge in molecular units, typically in the hundreds of millions. This enormous ratio of scales isolates theories of macroscopic phenomena from the details of the microscopic molecular world. A general meta-law of physics claims that the physical laws valid at one length scale are not very sensitive to the details of what happens at much smaller scales. Without this meta-law, physics would in fact be impossible, because we never know what lies below our currently deepest level of understanding.

The microscopic world impinges upon the macroscopic almost only through material constants, such as coefficients of elasticity and viscosity, characterizing the interactions between macroscopic amounts of matter. It is, of course, an important task for the physics of materials to derive the values of these constants, but this task lies outside the realm of continuum physics. It is nevertheless sometimes instructive to make simple models of the underlying atomic or molecular structure in order to obtain an understanding of the origin of macroscopic phenomena and of the limits to the continuum description.

This chapter paints in broad outline the transition from molecules to continuous matter, or mathematically speaking from point particles to fields. It is emphasized that the macroscopic continuum description must necessarily be statistical in nature, but that random statistical fluctuations are strongly suppressed by the enormity of the number of molecules in any macroscopic material object. The modern fields of nanophysics and biophysics straddle the border between the continuum and particle descriptions of matter, resulting in numerous new phenomena outside the scope of classical continuum physics. These topics will not be covered here. The central theme of this book is the recasting of Newton's laws for point particles into a systematic theory of continuous matter, and the application of this theory to the wealth of exotic and everyday phenomena in the macroscopic material world.



The quantitative meaning of a chemical formula. The boxes represent fixed amounts, for example moles.



Lorenzo Romano Amadeo Carlo Avogadro (1776–1856). Italian philosopher, lawyer, chemist, and physicist. Count of Quaregna and Cerratto. Formulated that equal volumes of gas contain equal numbers of molecules. Also argued that simple gases consist of diatomic molecules. (Source: Wikimedia Commons.)

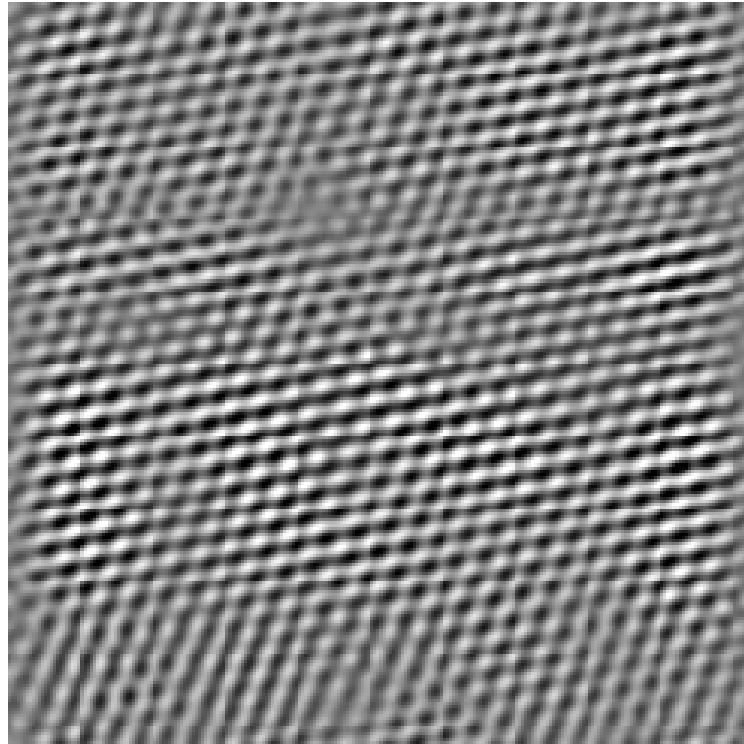


Figure 1.1. How continuous matter really looks at the atomic scale. Noise-filtered image of freshly cleaved Mica obtained by atomic force microscopy, approximately 225 Angstrom on a side. This granularity of matter is ignored in continuum physics. (Source: Mark J. Waner, PhD dissertation, Michigan State University, 1998. With permission.)

1.1 Molecules

Chemical reactions such as $2\text{H}_2 + \text{O}_2 \rightarrow 2\text{H}_2\text{O}$ are characterized by simple integer coefficients. Two measures of hydrogen plus one measure of oxygen yield two measures of water without anything left over of the original ingredients. What are these measures? For gases at the same temperature and pressure, a measure is simply a fixed volume, for example a liter, so that two liters of hydrogen plus one liter of oxygen yield two liters of water vapor, assuming that the water vapor without condensing can be brought to the same temperature and pressure as the gases had before the reaction. In 1811, Count Avogadro of Italy proposed that the simple integer coefficients in chemical reactions between gases could be explained by the rule that equal volumes of gases contain equal numbers of molecules (at the same temperature and pressure).

The various measures do not weigh the same. A liter of oxygen is roughly 16 times heavier than a liter of hydrogen at the same temperature and pressure. The mass of any amount of water vapor must—of course—be the sum of the masses of its ingredients, hydrogen and oxygen. The reaction formula tells us that two liters of water vapor weigh roughly $(2 \times 1) + (1 \times 16) = 18$ times a liter of hydrogen. Such considerations led to the introduction of the concept of relative molecular mass (or weight) in the ratio 1:16:9 for molecular hydrogen, molecular oxygen, and water. Today, most people would prefer to write these proportions as 2:32:18, reflecting the familiar molecular masses of H_2 , O_2 , and H_2O , respectively. In practice, relative molecular masses deviate slightly from integer values, but for the sake of argument we shall disregard that here.

Mole and molar mass

In the beginning there was no way of fixing an absolute scale for molecular mass. To define a scale that was practical for the chemist at work in his laboratory, the *molar mass* of atomic hydrogen (H) was arbitrarily set to be 1 gram. The ratios of molecular masses obtained from chemical reactions would then determine the molar mass of any other substance. Thus the molar mass of hydrogen gas (H₂) is 2 grams and that of oxygen gas (O₂) is 32 grams, whereas water (H₂O) has a molar mass of 18 grams because the chemical reaction tells us that $(2 \times 2) + (1 \times 32) = 2 \times 18$ grams. This system could be extended to all chemical reactions allowing the determination of molar mass for any substance participating in such processes.

An amount of a substance with mass equal to its molar mass is called a *mole* and the symbol used for the unit is mol. Thus 1 gram of atomic hydrogen, 2 grams of molecular hydrogen, 32 grams of molecular oxygen, or 18 grams of water all make up one mole. The chemical reaction formula $2\text{H}_2 + \text{O}_2 \rightarrow 2\text{H}_2\text{O}$ simply expresses that 2 moles hydrogen gas plus 1 mole oxygen gas produces 2 moles water. According to Avogadro's hypothesis, *the number of molecules in a mole of any substance is the same*, appropriately called Avogadro's number by Perrin and denoted N_A . In 1908 Perrin carried out the first modern determination of its value from Brownian motion experiments. Perrin's experiments relying on Einstein's recent (1905) theory of Brownian motion were not only seen as a confirmation of this theory but also as the most direct evidence for the reality of atoms and molecules.

Today, Avogadro's number is defined to be the number of atoms in exactly 12 grams of the fundamental carbon isotope ¹²C, which therefore has molar mass equal to exactly 12 g mol⁻¹. Avogadro's number is determined empirically, and the accepted 2006 value [1] is $N_A = 6.02214179(30) \times 10^{23}$ molecules per mole, with the parenthesis indicating the absolute error on the last digits.

Unit of mass: The definition of Avogadro's number depends on the definition of the unit of mass, the kilogram, which is (still) defined by a prototype from 1889 stored by the *International Bureau of Weights and Measures* near Paris, France. Copies of this prototype and balances for weighing them can be made to a precision of one part in 10⁹. Maybe already in 2011 a new definition of the kilogram will replace this ancient one [MMQ&06], for example by defining the kilogram to be the total mass of an exact number of ¹²C atoms. Avogadro's number will then also become an exact number without error.

Molecular separation length

Consider a sample of a pure substance with volume V and mass M . If the molar mass of the substance is denoted M_{mol} , the number of moles in the sample is $n = M/M_{\text{mol}}$, and the number of molecules $N = nN_A$. The volume per molecule is V/N , and a cube with this volume would have sides of length

$$L_{\text{mol}} = \left(\frac{V}{N}\right)^{1/3} = \left(\frac{M_{\text{mol}}}{\rho N_A}\right)^{1/3}, \quad (1.1)$$

where $\rho = M/V$ is the mass density. This *molecular separation length* sets the scale at which the molecular granularity of matter dominates the physics, and any conceivable continuum description of bulk matter must utterly fail.

For liquids and solids where the molecules touch each other, this length is roughly the size of a molecule. For solid iron we get $L_{\text{mol}} \approx 0.23$ nm, and for liquid water $L_{\text{mol}} \approx 0.31$ nm. Since by Avogadro's hypothesis equal gas volumes contain an equal number of molecules, the molecular separation length for *any* (ideal) gas at normal temperature and pressure ($p = 1$ atm and $T = 20^\circ\text{C}$) becomes $L_{\text{mol}} \approx 3.4$ nm. There is a lot of vacuum in a volume of gas, in fact about 1000 times the true volume of the molecules at normal temperature and pressure.

Jean-Baptiste Perrin (1870–1942). French physicist. Received the Nobel Prize for his work on Brownian motion in 1926. He founded several French scientific institutions, among them the now-famous *Centre National de la Recherche Scientifique (CNRS)*.

* **Mixtures**¹: The above expression for L_{mol} may also be used for a mixture of pure substances, provided M_{mol} is taken to be a suitable average over the molar masses M_i^{mol} of the i -th pure component (consisting of only one kind of molecules). For a mixture sample of mass M containing the mass M_i of each component, the total mass becomes the sum $M = \sum_i M_i$. The number of moles of the i -th component is $n_i = M_i/M_i^{\text{mol}}$ and the total number of moles in the sample is $n = \sum_i n_i$. Characterizing the composition of the mixture by the *molar fraction* $X_i = n_i/n$ of each component, the average molar mass, $M_{\text{mol}} = M/n$, becomes

$$M_{\text{mol}} = \sum_i X_i M_i^{\text{mol}}, \quad (1.2)$$

where $\sum_i X_i = 1$. If we instead describe the composition by the *mass fraction* $Y_i = M_i/M = X_i M_i^{\text{mol}}/M_{\text{mol}}$ of each component, and use that $\sum_i X_i = 1$, the average molar mass is determined by the reciprocal sum,

$$\frac{1}{M_{\text{mol}}} = \sum_i \frac{Y_i}{M_i^{\text{mol}}}, \quad (1.3)$$

where $\sum_i Y_i = 1$.

Dry air is a molar mixture of 78.08% nitrogen, 20.95% oxygen and 0.93% argon with an average molar mass of $M_{\text{mol}} = 28.95 \text{ g mol}^{-1}$. By mass the mixture is 75.56% nitrogen, 23.15% oxygen, and 1.29% argon, and has of course the same average molar mass.

Intermolecular forces and states of matter

Johannes Diederik van der Waals (1837–1923). Dutch physicist. Developed an equation of state for gases, now carrying his name. Received the Nobel Prize in 1910 for his work on liquids and gases.

Apart from the omnipresent gravitational interaction between all bodies, material interactions are entirely electromagnetic in nature, from the fury of a tornado to the gentlest kiss. A detailed understanding of the so-called van der Waals forces acting between neutral atoms and molecules falls outside the scope of this book. Generally, however, the forces are strongly repulsive if the molecules are forced closer than their natural sizes allow, and moderately attractive when they are moved apart. This tug of war between repulsion and attraction determines an equilibrium distance between them that is comparable to the molecular size. In Figure 1.2 is shown an example of such a potential, the famous Lennard-Jones potential (see Problem 1.2).

When huge numbers of molecules are put together to make up a body, they may arrange themselves in a number of different ways to minimize their total energy. The total energy receives negative contributions from the intermolecular potential energy, which attempts to bind the molecules to each other near equilibrium, and positive contributions from the kinetic energy in their thermal motion, which tends to make them fly apart. The three classic states of neutral matter—solid, liquid, and gas—depend, broadly speaking, on the competition between negative binding energy and positive thermal energy.

Solids: In *solid matter* the binding is so strong that thermal motion cannot overcome it. The molecules remain bound to each other by largely elastic forces, and constantly undergo small-amplitude thermal motion around their equilibrium positions. If increasing external forces are applied, solids will begin to *deform* elastically, until they eventually become plastic or even fracture. A solid body retains its shape independently of the shape of a container large enough to hold it, apart from small deformations, for example due to gravity.

Liquids: In *liquid matter* the binding is weaker than in solid matter, although it is still hard for a molecule on its own to leave the company of the others through an open liquid surface. The molecules stay in contact but are not locked to their neighbors. Molecular conglomerates may form and stay loosely connected for a while, as for example chains of water molecules.

¹The asterisk indicates that this part of the text can be skipped in a first reading.

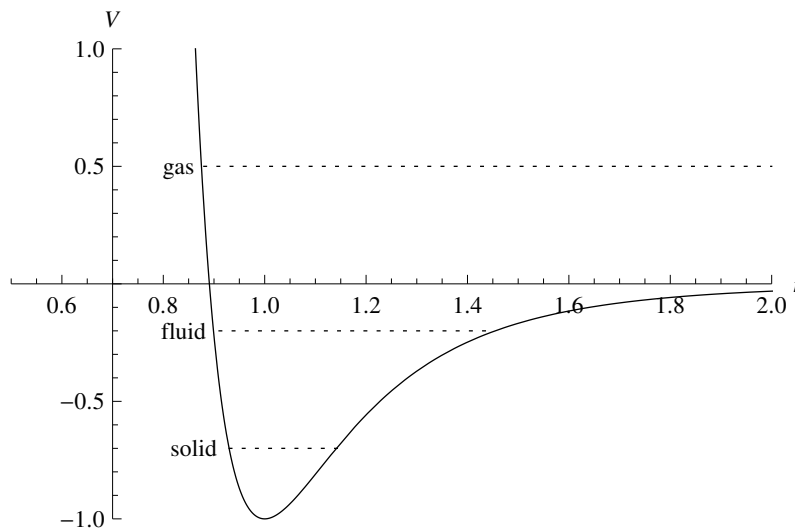


Figure 1.2. Sketch of the intermolecular potential energy $V(r)$ between two roughly spherical neutral molecules as a function of the distance r between their centers. It is attractive at moderate range and strongly repulsive at close distance. The equilibrium between attraction and repulsion is found at the minimum of the potential. Here the units are arbitrarily chosen such that the equilibrium distance becomes $r = 1$ and the minimum potential $V(1) = -1$. The horizontal dashed lines suggest the thermal energy levels for solid, liquid, and gaseous matter.

Under the influence of external forces, for example gravity, a liquid will undergo bulk motion, called *flow*, a process that may be viewed as a kind of continual fracturing. A liquid will not expand to fill an empty container completely, but will under the influence of external forces eventually adapt to its shape wherever it touches it.

Gases: In *gaseous matter* the molecules are bound so weakly that the thermal motion easily overcomes it, and they essentially move around freely between collisions. A gas will always expand to fill a closed empty container completely. Under the influence of external forces, for example a piston pushed into the container, a gas will quickly flow to adapt to the changed container shape.

Granular matter

Molecules and atoms represent without discussion the basic granularity of all macroscopic matter, but that does not mean that all macroscopic matter can be viewed as continuous. Barley grain, quartz sand, living beings, buildings, and numerous materials used in industry are examples of matter having a discrete, granular substructure that influences the material properties even at considerably larger length scales. The main difference between granular matter and molecular matter lies in the friction that exists between the granular elements. Although grain may flow like water in great quantities, the friction between the grains can make them stick to each other, forming a kind of solid that may block narrow passages. A heap of grain does not flatten out like a puddle of water, and neither do the wind-blown ripples of dry sand in the desert. You may also build a castle of wet sand, the main part of which remains standing even when dried out.

In this book we shall not consider granular matter as such, although at sufficiently large length scales, granular matter may in many respects behave like continuous matter, whether the grains are tiny as the quartz crystals in sand or enormous as the galaxies in the universe.

1.2 The continuum approximation

Classical thermodynamics comfortably deals with homogeneous bodies made from moles of matter where the molecular granularity can be safely ignored. Continuum physics on the other hand aims to describe bodies with spatial variations in thermodynamic quantities, such as density, pressure, and temperature. That aim immediately raises a conflict between the characteristic length scale for such variations and the number of molecules that are necessary to define the quantities. The more rapidly a quantity varies, the smaller volumes of matter must be considered and the more important becomes the molecular granularity.

Whether a given number of molecules is large enough to warrant the use of a smooth continuum description of matter depends on the desired precision. Since matter is never continuous at sufficiently high precision, continuum physics is always an approximation. But as long as the fluctuations in physical quantities caused by the discreteness of matter are smaller than the desired precision, matter may be taken to be continuous. To observe the continuity, one must so to speak avoid looking too sharply at material bodies. Fontenelle stated in a similar context that “*Science originates from curiosity and bad eyesight*”.

Here we shall only discuss the limits to the continuum approximation for gases. Similar limits exist for solids and liquids but are more difficult to estimate. The gas results may nevertheless be used as an upper limit to the fluctuations (see Section 23.3 on page 398).

Density fluctuations

Consider a fixed small volume V of a pure gas with molecules of mass m . If at a given time t the number of molecules in this volume is N , the mass density at this time becomes

$$\rho = \frac{Nm}{V}. \quad (1.4)$$

Due to rapid random motion of the gas molecules, the number N will be different at a later time $t + \Delta t$. Provided the time interval Δt is much larger than the time interval τ between molecular collisions, the molecules in the volume V will at the later time be an essentially random sample of molecules taken from a much larger region around V . The probability that any particular molecule from this larger region ends up in the volume V will be tiny. From general statistical considerations it follows that the root-mean-square size of the fluctuations in the number of molecules is given by $\bar{\Delta}N = \sqrt{N}$ (see Figure 1.3, and Problem 1.1). Since the density is linear in N , the relative fluctuation in density becomes

$$\frac{\bar{\Delta}\rho}{\rho} = \frac{\bar{\Delta}N}{N} = \frac{1}{\sqrt{N}}. \quad (1.5)$$

In classical macroscopic thermodynamics where typically $N \sim N_A$, the relative fluctuation becomes of magnitude 10^{-12} and can safely be ignored.

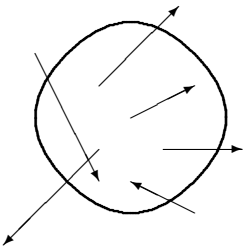
In continuum physics this is not so. If we want the relative density fluctuation to be smaller than a given value $\bar{\Delta}\rho/\rho \lesssim \epsilon$, we must require $N \gtrsim \epsilon^{-2}$. The smallest acceptable number of molecules, ϵ^{-2} , occupies a volume $\epsilon^{-2}L_{\text{mol}}^3$, where L_{mol} is the molecular separation length (1.1). A cubic cell with this volume has side length

$$L_{\text{micro}} = \epsilon^{-2/3}L_{\text{mol}}. \quad (1.6)$$

Thus, to secure a relative precision $\epsilon = 10^{-3}$ for the density, the microscopic cell should contain 10^6 molecules and have side length $L_{\text{micro}} = 100L_{\text{mol}}$. For an ideal gas under normal conditions we find $L_{\text{micro}} \approx 0.34 \mu\text{m}$.

The micro scale diverges for $\epsilon \rightarrow 0$, substantiating the claim that *it is impossible to maintain a continuum description to arbitrarily high precision*.

Bernard le Bovier de Fontenelle (1657–1757). French intellectual, poet, author, and philosopher. Member of both the French Academy of Letters and of Sciences. Made popular accounts of the theories of René Descartes, whom he admired greatly.



In a gas the molecules move rapidly in and out of a small volume with typical velocities comparable to the speed of sound.

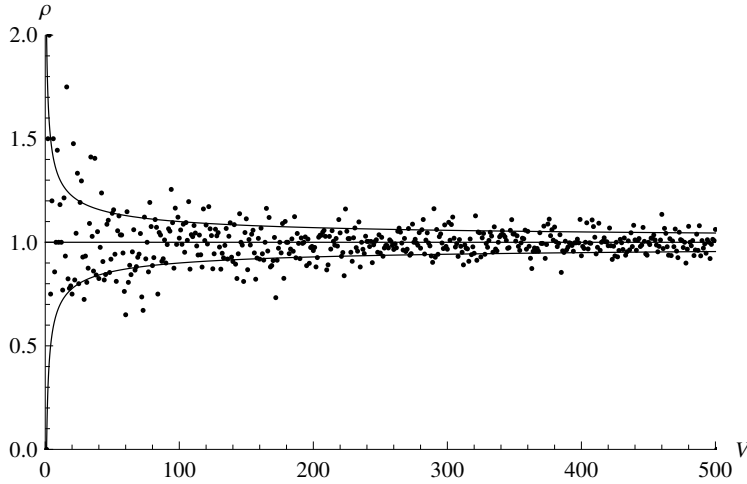


Figure 1.3. Simulation of density fluctuations as a function of volume size. A three-dimensional “universe” consisting of $20 \times 20 \times 20 = 8,000$ cells of unit volume is randomly filled with as many unit mass “molecules”. On average each of the 8,000 cells should contain a single molecule, corresponding to a density of $\rho = 1$. A “volume” consisting of V cells will not contain precisely V molecules, and thus has a density that deviates from unity. The plot shows the density of a random collection of V cells as a function of V . The drawn curves, $\rho = 1 \pm 1/\sqrt{V}$, indicate the expected fluctuations.

Macroscopic smoothness

As a criterion for a smooth continuum description we demand that the relative variation in density between neighboring cells should be less than the desired precision ϵ . The change in density between the centers of neighboring cells along some direction x is of magnitude $\Delta\rho \approx L_{\text{micro}} |\partial\rho/\partial x|$. Demanding the relative variation to be smaller than the precision, $\Delta\rho/\rho \lesssim \epsilon$, we obtain a constraint on the magnitude of the density derivative,

$$\left| \frac{\partial\rho}{\partial x} \right| \lesssim \frac{\rho}{L_{\text{macro}}}, \quad (1.7)$$

where

$$L_{\text{macro}} = \epsilon^{-1} L_{\text{micro}}. \quad (1.8)$$

As long as the above condition is fulfilled, the density may be considered to vary smoothly, because the density changes over the micro-scale are imperceptible. Any significant change in density must take place over distances larger than L_{macro} . With $\epsilon = 10^{-3}$ we find $L_{\text{macro}} \approx 1000 L_{\text{micro}} \approx 0.34$ mm for an ideal gas under normal conditions.

The thickness of interfaces between macroscopic bodies is typically on the order of L_{mol} and thus much smaller than L_{macro} . Consequently, these regions of space fall outside the smooth continuum description. In continuum physics, interfaces appear instead as surface discontinuities in the otherwise smooth macroscopic description of matter.

Velocity fluctuations

Everyday gas speeds are small compared to the molecular velocities—unless one is traveling by jet aircraft or cracking a whip. What we normally mean by wind is the bulk drift of air, not the rapid molecular motions. So even if the individual molecules move very fast in random directions, the center of mass of a collection of N molecules in a small volume V will normally move with much slower velocity, which for large N approximates the drift speed v .

Gases consist mostly of vacuum, and apart from an overall drift, the individual molecules move in all possible directions with average root-mean-square speed (see page 24),

$$v_{\text{mol}} = \sqrt{\frac{3R_{\text{mol}}T}{M_{\text{mol}}}}, \quad (1.9)$$

where $R_{\text{mol}} = 8.31447 \text{ J K}^{-1} \text{ mol}^{-1}$ is the universal molar gas constant and T the absolute temperature. For air at normal temperature one finds $v_{\text{mol}} \approx 500 \text{ m s}^{-1}$.

Under very general assumptions the root-mean-square fluctuation in the center-of-mass speed is $\overline{\Delta v} = v_{\text{mol}}/\sqrt{N}$ (see Problem 1.3). Since the fluctuation scale is set by the molecular velocity, it takes much larger numbers of molecules to be able ignore the fluctuations in everyday gas velocities. To maintain a relative precision ϵ in the velocity fluctuations we must require $\overline{\Delta v}/v \lesssim \epsilon$, implying that the linear size of a gas volume must be larger than

$$L'_{\text{micro}} = \left(\frac{v_{\text{mol}}}{v}\right)^{2/3} L_{\text{micro}}. \quad (1.10)$$

The velocity fluctuations of a gentle steady wind, say $v \approx 0.5 \text{ m s}^{-1}$, can be ignored with precision $\epsilon \approx 10^{-3}$ for volumes of linear size larger than $L'_{\text{micro}} = 100L_{\text{micro}} \approx 34 \mu\text{m}$. The smoothness scale should similarly be $L'_{\text{macro}} = \epsilon^{-1}L'_{\text{micro}}$, which in this case becomes $L'_{\text{macro}} \approx 34 \text{ mm}$. A hurricane wind, $v \approx 50 \text{ m s}^{-1}$, only requires volumes of linear size $L'_{\text{micro}} = 4.6L_{\text{micro}} \approx 1.6 \mu\text{m}$ to yield the desired precision, but in this case fluctuations due to turbulence will anyway completely swamp the molecular fluctuations.

* Mean free path

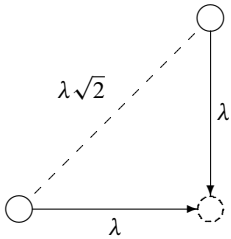
Another condition for obtaining a valid continuum description is that molecules should interact with each other to “iron out” strong differences in velocities. If there were no interactions, a molecule with a given velocity would keep on moving with that velocity forever. In solids and liquids where the molecules are closely packed, these interactions take place over a molecular separation length.

In a gas every molecule traces out a straight path through the vacuum until it collides with another molecule. The *mean free path* λ is defined as the average distance traveled by a single molecule. The other molecule is, however, not a “sitting duck” but travels itself on average also a distance λ before it is hit by the first. Since the movement directions are arbitrary, the second molecule “sees” the first coming toward it along a straight line with a speed that on average is orthogonal to the first and thus $\sqrt{2}$ times the root-mean-square speed v_{mol} . Denoting the molecular diameter by d , the collision will only happen if the center of the second molecule is within a “striking distance” d from the path of the first, that is, inside a cylinder of radius d . There is on average one molecule in the molecular volume L_{mol}^3 , so the mean free path is defined such that the average volume swept out by the moving spheres equals the molecular volume, or $\pi d^2 \sqrt{2} \lambda = L_{\text{mol}}^3$. The mean free path thus becomes

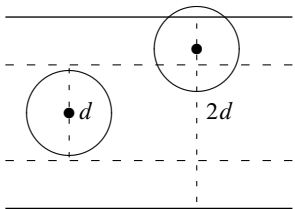
$$\lambda = \frac{L_{\text{mol}}^3}{\sqrt{2} \pi d^2} = \frac{M_{\text{mol}}}{\sqrt{2} \pi d^2 \rho N_A}. \quad (1.11)$$

Since $\lambda/L_{\text{mol}} \sim \rho^{-2/3}$, the mean free path will in sufficiently dilute gases always become larger than the micro scale and should instead be used to define the smallest linear scale for the continuum description.

Air consists mainly of nitrogen and argon molecules with an average molar mass $M_{\text{mol}} \approx 29 \text{ g mol}^{-1}$ and average diameter $d \approx 0.37 \text{ nm}$ [2]. At normal temperature and pressure the mean free path becomes $\lambda \approx 65 \text{ nm}$, which is five times smaller than the microscopic length scale for the density, $L_{\text{micro}} \approx 340 \text{ nm}$ (for $\epsilon = 10^{-3}$). The mean collision time may be estimated as $\tau \approx \lambda/v_{\text{mol}}$, and becomes for air $\tau \approx 0.13 \text{ ns}$.



Two molecules on collision course with equal orthogonal velocities. Each moves the distance λ between collisions, and thus $\lambda\sqrt{2}$ relative to each other.



A sphere of diameter d will collide with any other sphere of the same diameter with its center inside a cylinder of diameter $2d$.

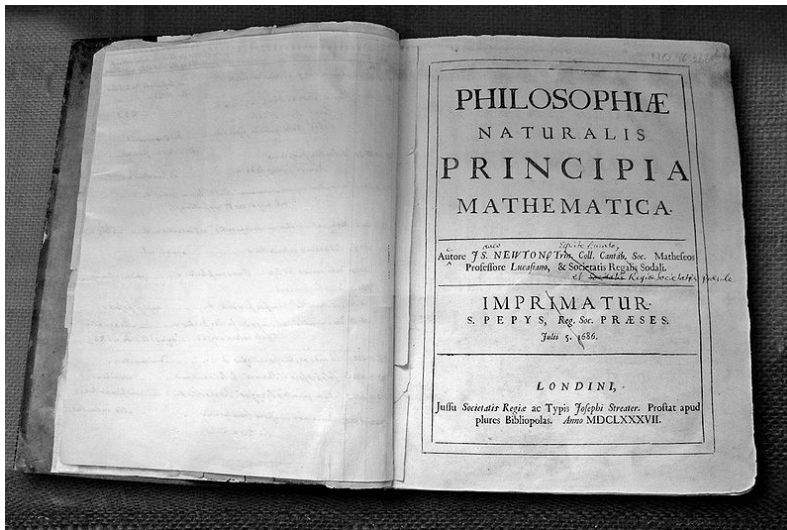


Figure 1.4. Newton's own copy of *Principia* with handwritten corrections for the second edition. (Source: Andrew Dunn (2004). Wikimedia Commons.)

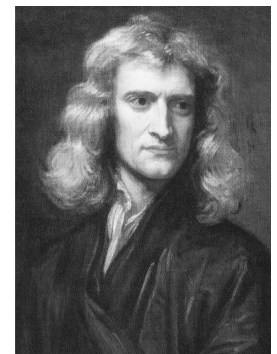
1.3 Newtonian mechanics

In Newtonian mechanics the elementary material object is a point particle (“molecule”) with a fixed mass. Newton postulated three laws for such particles, which formed the basis for rational mechanics in the following centuries and became a role model for all other natural sciences. It is not the intention here to enter into a discussion of the consistency of these laws or other objections that can be raised, but just to present them in a short form supposed to be suitable for any reader of this book. Here they are:

1. *There exist (inertial) reference frames in which a particle moves with constant velocity along a straight line when it is not acted upon by any forces.*
2. *The mass times the acceleration of a particle equals the sum of all forces acting on it.*
3. *If a particle acts on another with a certain force, the other particle acts back on the first with an equal and opposite force.*

In Appendix A you will find a little refresher course in Newtonian mechanics.

Newton's Second Law is *the* fundamental equation of motion. Mathematically, this law is expressed as a second-order differential equation in time. Since the force acting on any given particle can depend on the positions and velocities of the particle itself and of other particles as well as on external parameters, the dynamics of a collection of particles becomes a web of coupled ordinary second-order differential equations in time. Even if macroscopic bodies *are* huge collections of atoms and molecules, it is completely out of the question to solve the resulting web of differential equations. In addition, there is the problem that molecular interactions are quantum mechanical in nature, so that Newtonian mechanics does not apply at the atomic level. This knowledge is, however, relatively new and has as mentioned earlier some difficulty in making itself apparent at the macroscopic level. So although quantum mechanics definitely rules the world of atoms, its special character is rarely amplified to macroscopic proportions, except in low-temperature phenomena such as superconductivity and superfluidity.



Sir Isaac Newton (1643–1727). English physicist and mathematician. Founded classical mechanics on three famous laws in his books *Philosophiæ Naturalis Principia Mathematica* (1687). Newton developed calculus to solve the equations of motion, and formulated theories of optics and of chemistry. He still stands as perhaps the greatest scientific genius of all time. (Source: Wikimedia Commons.)

Laws of balance

In Newtonian particle mechanics, a “body” is taken to be a fixed collection of point particles with unchangeable masses, each obeying the Second Law. For any body one may define various *global mechanical quantities*, which like the total mass are calculated as sums over contributions from each and every particle in the body. Some of the global quantities are *kinematic*: momentum, angular momentum, and kinetic energy. Others are *dynamic*: force, moment of force, and power (rate of work of all forces).

Newton’s Second Law for particles leads to three simple *laws of balance*—often called conservation laws—relating the kinematic and dynamic quantities. They are (in addition to the trivial statement that the mass of a Newtonian body never changes):

- *The rate of change of momentum equals force,*
- *The rate of change of angular momentum equals moment of force,*
- *The rate of change of kinetic energy equals power.*

Even if these laws are insufficient to determine the dynamics of a multi-particle body, they represent seven individual constraints (two vectors and one scalar) on the motion of any collection of point particles, regardless of how complex it is.

In continuum mechanics any volume of matter may be considered to be a body. As the dynamics unfolds, matter is allowed to be exchanged between the environment and the body, but we shall see that *the laws of balance can be directly carried over to continuum mechanics when exchange of matter between a body and its environment is properly taken into account*. Combined with simplifying assumptions about the macroscopic behavior, for example symmetry, the laws of balance for continuous matter also turn out to be quite useful for obtaining quick solutions to a variety of problems.

Material particles

In continuum physics we generally speak about *material particles* as the elementary constituents of continuous matter, obeying Newton’s equations. A material particle will always contain a large number of molecules but may in the continuum description be thought of as infinitesimal or point-like. From the preceding analysis we know that material particles cannot be truly infinitesimal, but represent *the smallest bodies that may consistently be considered part of the continuum description within the required precision*. Continuum physics does not “on its own” go below the level of the material particles. Even if the mass density may be determined by adding together the masses of all the molecules in a material particle and dividing the sum by the volume of the particle, this procedure falls, strictly speaking, outside continuum physics.

Although we normally think of material particles as being identical in different types of matter, it is sometimes necessary to go beyond the continuum approximation and look at their differences. In solids, we may with some reservation think of *solid particles* as containing a fixed collection of molecules, whereas in liquids and especially in gases we should not forget that the molecules making up a *fluid particle* at a given instant will shortly be replaced by other molecules. If the molecular composition of the matter in the environment of a material particle has a slow spatial variation, this incessant molecular game of “musical chairs” may slowly change the composition of the material inside the particle. Such *diffusion processes* driven by spatial variations in material properties lie at the very root of fluid mechanics. Even a spatial velocity variation will drive momentum diffusion, causing internal (viscous) friction in the fluid.

1.4 Reference frames

Physics is a quantitative discipline using mathematics to relate measurable quantities expressed in terms of real numbers. In formulating the laws of nature, undefined mathematical primitives—for example the points, lines and circles of Euclidean geometry—are not particularly useful, and such concepts have for this reason been eliminated and replaced by numerical representations everywhere in physics. This step necessitates a specification of the practical procedures by which these numbers are obtained in an experiment; for example, which units are being used.

Behind every law of nature and every formula in physics, there is a framework of procedural descriptions, a *reference frame*, supplying an operational meaning to all physical quantities. Part of the art of doing physics lies in comprehending this, often tacitly understood, infrastructure to the mathematical formalism. The reference frame always involves physical objects—balances to measure mass, clocks to measure time and rulers to measure length—that are not directly a part of the mathematical formalism. Precisely because they *are* physical objects, they can at least in principle be handed over or copied, and thereby shared among experimenters. This is in fact still done for the unit of mass (see page 3).

Agreement on a common reference frame for all measurable quantities is a necessary condition for doing science. The system of units, the *Système Internationale (SI)*, is today fixed by international agreement; but even if our common frame of reference for units is thus defined by social convention, physics is nevertheless objective. In principle our frames of reference could be shared with all other beings in the universe, or alternatively given a precise translation to theirs.

Time

Time is the number you read on your *clock*. There is no better definition. Clocks are physical objects that may be shared, compared, copied, and synchronized to create an objective meaning of *time*. Most clocks, whether they are grandfather clocks with a swinging pendulum or oscillating quartz crystals, are based on periodic physical systems that return to the same state again and again. Time intervals are simply measured by counting periods. There are also aperiodic clocks, for example hour glasses, and clocks based on radioactive elements. It is especially the latter that allow time to be measured on geological time scales. On extreme cosmological time scales the very concept of time becomes increasingly more theory laden; see for example [RZ09].

Like all macroscopic physical systems, clocks are subject to small fluctuations in the way they run. Some clocks are considered better than others because they keep time more stably with respect to copies of themselves as well as with clocks built on other principles. Grandfather clocks are much less stable than mechanical maritime chronometers that in turn are less stable than modern quartz clocks. The international frame of reference for time is always based on the most stable clocks currently available.

Unit of time: The unit of time, the *second*, was formerly defined as $1/86,400$ of a mean solar day. But the Earth's rotation is not that stable, and since 1966 the second has been defined by international agreement as the duration of 9,192,631,770 oscillations of the microwave radiation absorbed in a certain hyperfine transition in cesium-133, a metal that can be found anywhere on Earth [1]. A beam of cesium-133 atoms is used to stabilize a quartz oscillator at the right frequency by a resonance method, so what we call an atomic clock is really an atomically stabilized quartz clock. The intrinsic precision in this time standard has been continually improving and is now about 4×10^{-16} corresponding to about 1 second in 80 million years [LHJ07].

In the extreme mathematical limit, time may be taken to be a real number, and in Newtonian physics its value is assumed to be universally knowable.

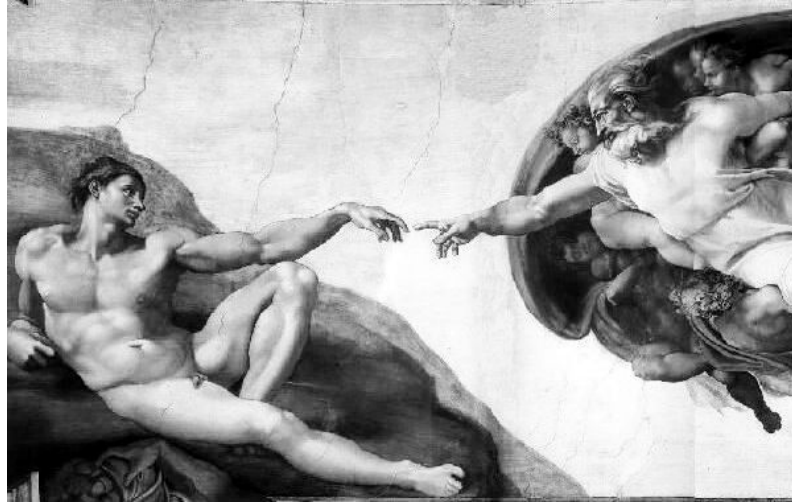


Figure 1.5. Independently of how different their reference frames, two observers who agree on the unique reality of any point in space can in principle determine the coordinate transformation relating them by listing their respective coordinates for each and every point in space. (Source: Fragment of “The Creation of Adam”, by Michelangelo Buonarroti (1511). Wikimedia Commons.)

Space

It is a mysterious and so far unexplained fact that physical *space* has three dimensions, which means that it takes exactly three real numbers to locate a point in space. These numbers are called the *coordinates* of the point, and the reference frame for coordinates is called the *coordinate system*. It must contain all the operational specifications for locating a point given the coordinates, and conversely obtaining the coordinates given the location. In this way we have relegated all philosophical questions regarding the *real* nature of points and of space to the operational procedures contained in the reference frame.

Earth coordinates: On Earth everybody navigates by means of a geographical coordinate system agreed upon by international convention, in which a point is characterized by latitude δ , longitude λ , and elevation h . Latitude and longitude are angles fixed by the Earth’s rotation axis and the position of the former Royal Observatory in Greenwich near London (UK). Elevation is defined as the signed height above the average sea level. The modern Global Positioning System (GPS) uses instead “fixed points in the sky” in the form of at least 24 satellites, and the geographical coordinates of any point on Earth as well as the absolute time in this point is determined from radio signals received from four or more of these satellites.

The triplet of coordinates that locates a point in a particular coordinate system is called its *position* in that coordinate system, and usually marked with a single symbol printed in boldface², for example $\mathbf{x} = (x_1, x_2, x_3)$. Given the position $\mathbf{x} = (x_1, x_2, x_3)$ of a point in one coordinate system, we now require that the position $\mathbf{x}' = (x'_1, x'_2, x'_3)$ of the exact same point in another coordinate system must be calculable from the first:

$$x'_1 = f_1(x_1, x_2, x_3), \quad x'_2 = f_2(x_1, x_2, x_3), \quad x'_3 = f_3(x_1, x_2, x_3). \quad (1.12)$$

More compactly this may be written $\mathbf{x}' = \mathbf{f}(\mathbf{x})$.

The postulate that there should exist a unique, bijective transformation connecting any given pair of coordinate systems reflects that *physical reality is unique* (see Figure 1.5), and

²The printed boldface notation is hard to reproduce in calculations with pencil on paper, so other graphical means are commonly used, for example a bar (\bar{x}), an arrow (\vec{x}), or underlining (\underline{x}).

Points may be visualized as dots on a piece of paper. Each point is labeled by its position in the chosen coordinate system (not visualized here).

• a
• b
• x

• $a \leftrightarrow a'$
• $b \leftrightarrow b'$
• $x \leftrightarrow x'$

In different coordinate systems the same points have different coordinates, connected by a transformation $\mathbf{x}' = \mathbf{f}(\mathbf{x})$.

that *different coordinate systems are just different ways of representing the same physical space in terms of real numbers.*

What's in a symbol?: There is nothing sacred about the symbols used for the coordinates. Mostly the coordinates are given suggestive or conventional symbolic names in particular coordinate systems, for example δ , λ , and h for the Earth coordinates above. In physics the familiar Cartesian coordinates are often denoted x , y , and z , while in more formal arguments one may retain the index notation, x_1 , x_2 , and x_3 . Cylindrical coordinates are denoted r , ϕ , and z , and spherical coordinates r , θ , and ϕ . These three coordinate systems are in fact the only ones to be used in this book.

Length

From the earliest times humans have measured the *length* of a road between two points, say a and b , by counting the number of steps it takes to walk along this road. This definition of length depends, however, on how you are built. In order to communicate to others the length of a road, the count of steps must be accompanied by a clear definition of the length of a step in terms of an agreed-upon unit of length.

Unit of length: Originally the units of length—inch, foot, span, and fathom—were directly related to the human body, but increasing precision in technology demanded better-defined units. In 1793 the meter was introduced as a ten millionth of the distance from equator to pole on Earth, and until far into the twentieth century a unique “normal meter” was stored in Paris, France. In 1960 the meter became defined as a certain number of wavelengths of a certain spectral line in krypton-86, an isotope of a noble gas that can be found anywhere on Earth. Since 1983 the meter has been defined by international convention to be the distance traveled by light in exactly $1/299,792,458$ of a second [1]. The problem of measuring lengths has thus been transferred to the problem of measuring time, which makes sense because the precision of the time current standard is at least a thousand times better than any proper length standard.

This method for determining the length of a path may be refined to any desired practical precision by using very short steps. In the extreme mathematical limit, the steps become infinitesimally small, and the road becomes a continuous path.

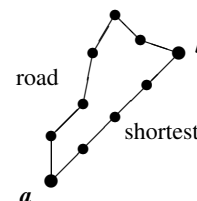
Distance

The shortest path between two points is called a *geodesic* and represents the “straightest line” between the points. In Euclidean space, a geodesic is indeed what we would intuitively call a straight line. On the spherical surface of the Earth geodesics are great circles, and airplanes and ships travel along them for good reason. The *distance* between two points is defined to be the length of a geodesic connecting them. Since the points are completely defined by their coordinates a and b in the chosen coordinate system, the distance must be a real positive function $d(a, b)$ of the two sets of coordinates.

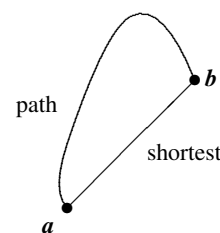
It is clear that the distance between two points must be the same in all coordinate systems, because it can, in principle, be determined by laying out rulers or counting steps between two points without any reference to coordinate systems. *Distance is a property of space rather than a property of the coordinate system.* The actual distance function $d'(a', b')$ in a new coordinate system may be different from the old, $d(a, b)$, but the numerical values have to be the same,

$$d'(a', b') = d(a, b), \quad (1.13)$$

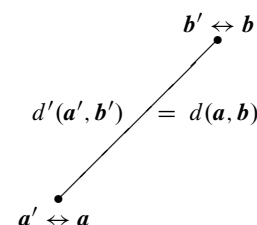
where $a' = f(a)$ and $b' = f(b)$ are calculated by the coordinate transformation (1.12). Knowing the distance function $d(a, b)$ in one coordinate system, it may be calculated in any other coordinate system by means of the appropriate coordinate transformation.



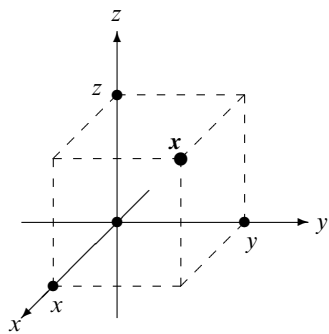
The length of the road between the positions a and b is measured by counting steps along the road. Different roads have different lengths, but normally there is a unique shortest road.



In the mathematical limit the shortest continuous path connecting a and b is called the geodesic. Normally, there is only one geodesic between any two points.

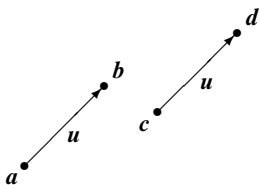


The distance is invariant under coordinate transformations.



A Cartesian coordinate system, here with coordinates labeled x , y , and z . Looks just like the ones you got to know (and love) in high school.

René Descartes (1596–1650). French scientist and philosopher, father of analytic geometry. Developed a theory of mechanical philosophy, later to be superseded by Newton’s work. Confronted with doubts about reality, he saw thought as the only argument for existence: “I think, therefore I am”.



The vector \mathbf{u} connects the point \mathbf{a} with the point \mathbf{b} . The same vector also connects the points \mathbf{c} and \mathbf{d} .

William Rowan Hamilton (1805–1865). Irish mathematical physicist. Created the Hamiltonian formulation of classical mechanics around 1833. Introduced dot- and cross-products of vectors and applied vector analysis to physical problems around 1845.

1.5 Cartesian coordinate systems

The space in which we live is nearly *flat* everywhere. Its geometry is Euclidean, meaning that Euclid’s axioms and the theorems deduced from them are valid everywhere. After Einstein we know, however, that space is not perfectly flat. In the field of gravity from a massive body, space necessarily curves, but normally only very little. At the surface of the Earth the radius of curvature of space due to Earth’s gravity is comparable to the distance to the Sun. The kind of physics that is the subject of this book may always be assumed to take place in perfectly flat Euclidean space.

The Cartesian distance function

In Euclidean geometry one derives Pythagoras’ theorem, which relates the lengths of the sides of any right-angled triangle. The simplicity of Pythagoras’ theorem favors the use of right-angled *Cartesian coordinate systems* in which the distance between two points in space is the square root of the sum of the squares of their coordinate differences.

$$d(\mathbf{a}, \mathbf{b}) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}. \quad (1.14)$$

Cartesian coordinates provide by far the most compact description of the geometry of flat space, and we shall in this book systematically describe physics “through the eyes” of Cartesian coordinates—even when we employ cylindrical or spherical coordinates.

Cartesian vectors

Since the distance between two points only depends on coordinate differences, triplets of coordinate differences naturally play a major role in Cartesian coordinate systems. Such triplets are called *vectors* and are marked with boldface in the same way as positions, for example $\mathbf{u} = (u_1, u_2, u_3) = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$, which is constructed from the coordinate differences of the positions \mathbf{b} and \mathbf{a} . Geometrically, this vector may be visualized as an arrow connecting \mathbf{a} to \mathbf{b} . As there is no “memory” in a vector about the absolute positions of the points, the same vector \mathbf{u} will *carry*³ you from any given position to another, provided the positions have the same coordinate difference. A position triplet \mathbf{x} connects the origin of the coordinate system $\mathbf{0} = (0, 0, 0)$ to the particular point in space located by \mathbf{x} , and positions may for this reason also be viewed as vectors (in Cartesian coordinates⁴). The mathematical concept of a vector is usually attributed to W. R. Hamilton.

The structure of the Cartesian distance function makes it natural to define linear operations on vectors, such as multiplication with a constant, $k\mathbf{u} = (ku_1, ku_2, ku_3)$; addition, $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$; and subtraction, $\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3)$. Mathematically, this makes the set of all vectors a three-dimensional vector space. Other operations, such as the dot-product, $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$, and the cross-product, $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$, may also be defined and have simple geometric interpretations.

In Appendix B Cartesian vector and tensor algebra is set up and analyzed in full detail. Most readers should already have met and worked with geometric vectors and will only need to consult this appendix in rare cases, for example to get a proper understanding of tensors and coordinate transformations. Coordinate transformations are central to analytic geometry and lead to the characterization of geometric quantities—scalars, vectors, and tensors—by the way they transform rather than in often ill-defined geometric terms.

³The word “vector” is Latin for “one who carries”, derived from the verb “vehere”, meaning to carry, and also known from “vehicle”. In epidemiology a “vector” is a carrier of disease.

⁴In non-Euclidean spaces or in curvilinear coordinate systems, the vector concept is only meaningful for infinitesimal coordinate differences in the infinitesimal neighborhood of any point (the local Euclidean tangent space).

1.6 Fields

In the extreme mathematical limit, material particles are taken to be truly infinitesimal and all physical properties of the particles as well as the forces acting on them are described by smooth—or at least piecewise smooth—functions of space and time. Continuum physics is therefore a theory of *fields*. Mathematically, a field f is simply a real-valued function, $f(\mathbf{x}, t) = f(x_1, x_2, x_3, t)$ or $f(\mathbf{x}, t) = f(x, y, z, t)$, of the spatial coordinates \mathbf{x} and time t , representing the value of a physical quantity in this point of space at this time⁵.

English	<i>field</i>
German	<i>feld</i>
Dutch	<i>veld</i>
Danish	<i>felt</i>
Swedish	<i>fält</i>
French	<i>champ</i>
Italian	<i>campo</i>
Spanish	<i>campo</i>
Russian	<i>polje</i>

The fields of continuum physics

We have already met the mass density field $\rho(\mathbf{x}, t)$. Knowing this field, the mass dM of a material particle occupying the volume dV near the point \mathbf{x} at time t can be calculated as

$$dM = \rho(\mathbf{x}, t) dV. \quad (1.15)$$

We shall mostly suppress the explicit space and time variables and just write $dM = \rho dV$.

Sometimes a collection of functions is also called a field and the individual real-valued members are called its *components*. The most fundamental field of fluid mechanics, the velocity field $\mathbf{v} = (v_1, v_2, v_3)$ or $\mathbf{v} = (v_x, v_y, v_z)$, has three components, one for each of the Cartesian coordinate directions. The velocity field $\mathbf{v}(\mathbf{x}, t)$ determines the momentum,

$$d\mathcal{P} = \mathbf{v}(\mathbf{x}, t) dM, \quad (1.16)$$

of a material particle of mass dM near \mathbf{x} at time t . The velocity field will be of major importance in formulating the dynamics of continuous systems.

Besides fields characterizing the state of the material, such as mass density and velocity, it is convenient to define fields that characterize the forces acting on and within the material. The gravitational acceleration field $\mathbf{g}(\mathbf{x}, t)$ penetrates all bodies from afar and acts on a material particle of mass dM with a force

$$d\mathcal{F} = \mathbf{g}(\mathbf{x}, t) dM. \quad (1.17)$$

Using that $dM = \rho dV$ we may also write gravity as a *body force* (or *volume force*) of the form, $d\mathcal{F} = \mathbf{f} dV$, with a density of force $\mathbf{f} = \rho\mathbf{g}$. It has infinite range, and the same is true for electromagnetism, but that also ends the list. No other forces in nature seem to have infinite range.

Some force fields are only meaningful for regions of space where matter is actually present, as for example the density ρ or the pressure field p , which acts across the imagined contact surfaces that separate neighboring volumes of a fluid at rest. Pressure is, however, not the only *contact force*. Fluids in motion, solids and more general materials, have more complicated contact forces that can only be fully described by the nine-component stress field, $\boldsymbol{\sigma} = \{\sigma_{ij}\}$, which is a (3×3) matrix field with rows and columns labeled by Cartesian coordinates: $i, j = 1, 2, 3$ or $i, j = x, y, z$.

Mass density, velocity, gravity, pressure, and stress are the usual fields of continuum mechanics and will all be properly introduced in the chapters to come. Other fields are thermodynamic, like the temperature T , the specific internal energy U , or the specific entropy S . Some describe different states of matter, for example the electric charge density ρ_e and current density \mathbf{j}_e together with the electric and magnetic field strengths, \mathbf{E} and \mathbf{B} . Like gravity \mathbf{g} , these force fields are thought to exist in regions of space completely devoid of matter.

⁵In mathematics the tendency is to use coordinates labeled by integers. In physics we shall—as mentioned before—mostly label the coordinate axes x , y , and z , and use these labels as vector indices $\mathbf{u} = (u_x, u_y, u_z)$. The general position is denoted $\mathbf{x} = (x, y, z)$ and becomes admittedly a bit inconsistent because of the strange relations, $x_x = x$, $x_y = y$ and $x_z = z$. In many physics texts the general position is instead denoted $\mathbf{r} = (x, y, z)$, but that comes with its own esthetic problems in more formal analysis. There seems to be no easy way to get the best of both worlds.

The use of the word “field” in physics to denote a function of the spacetime coordinates has an unclear origin. The original meaning of phrases such as “gravitational field”, “electric field”, or “magnetic field” was presumably to denote regions of gravitational, electric, or magnetic influences in the otherwise empty space around a body. The meaning was later shifted to the mathematical representation of the strength (and direction) of such influences in every point of space.

There are also fields that refer to *material properties*, for example the coefficient of shear elasticity μ of a solid and the coefficient of shear viscosity η of a fluid. Such fields are often nearly constant within homogeneous bodies, that is, independent of space and time, and are mostly treated as *material constants* rather than true fields.

Field equations

Like all physical variables, fields evolve with time according to dynamical laws, called *field equations*. In continuum mechanics, the central equation of motion descends directly from Newton's Second Law applied to every material particle. Mass conservation, which is all but trivial and most often tacitly incorporated in particle mechanics, turns in continuum theory into an equation of motion for the mass density. Still other field equations such as Maxwell's equations for the electromagnetic fields have completely different and non-mechanical origins, although they do couple to the mechanical equations of motion via the Lorentz force.

Mathematically, field equations are *partial differential equations* in both space and time. This makes continuum mechanics considerably more difficult than particle mechanics where the equations of motion are ordinary differential equations in time. On the other hand, this greater degree of mathematical complexity also leads to a plethora of new and sometimes quite unexpected phenomena. Mathematically, field equations in three-dimensional space would be quite cumbersome to deal with, were it not for an efficient extension of vector methods to what is now called *vector calculus* or *field calculus*. It is introduced along the way in the chapters to come and presented with some rigor in Appendix C.

In some field theories, for example Maxwell's electromagnetism, the field equations are *linear* in the fields, but that is not the case in continuum mechanics. The *non-linearity* of the field equations of continuum mechanics is caused by the velocity field, which behaves like a "wind" that carries other fields (and itself) along in the motion. This adds a further layer of mathematical difficulty to this subject, making it very different from linear theories—and much richer. The non-linearity leads to dynamic instabilities and gives rise to the chaotic and as yet not fully understood phenomenon of *turbulence*, well known from our daily dealings with water and air.

Local versus global descriptions

In modern textbooks on continuum physics there has been a tendency to avoid introducing the physical concept of a material particle. Instead these presentations rely on the Newtonian global laws of balance to deduce the local continuum description—in the form of partial differential equations—by purely mathematical means. Although quite elegant and apparently free of physical interpretation problems, such an approach unfortunately obscures the conditions under which the local laws may be assumed to be valid.

In this book the concept of a material particle has been carefully introduced in the proper physical context set by the desired precision of the continuum description. The advantage of such an approach is that the local description of continuous systems in terms of partial differential equations in space and time may be interpreted as representing the Newtonian laws applied to individual material particles. Furthermore, this "materialistic" approach allows us to set physical limits to the validity of the partial differential equations involving local quantities such as the density and the velocity. Such quantities cannot be assumed to be physically meaningful at distance scales smaller than the microscopic length scale L_{micro} . Furthermore, to maintain a continuum description, major spatial changes in these quantities should not take place in regions smaller than the macroscopic length scale L_{macro} , a condition that limits the magnitude of the spatial derivatives found in partial differential equations. Through the dynamical equations expressed as partial differential equations, the spatial limits imposed by precision also set limits on the magnitude of the partial time derivatives.

Mathematically, the local and global equations are equivalent, and must both be presented in any textbook, including this one. The local equations allow us to find exact solutions, either analytically or numerically, while the global equations are well suited for getting approximate solutions and making estimates.

Physical reality of force fields

Whereas the mass density and the pressure only have physical meaning in regions actually containing matter (but may be defined to be zero in vacuum), the gravitational field is assumed to exist and take non-vanishing values even in the vacuum. It specifies the force that would be exerted on a unit mass particle at a given point, but the field is assumed to be there even if no particles are present. In non-relativistic Newtonian physics, the gravitational field has no independent physical meaning and may be completely eliminated and replaced by non-local forces acting between material bodies. The true physical objects appear to be the material bodies, and the gravitational field is just a mathematical convenience for calculating the gravitational forces exerted by and on these bodies according to Newton's law of gravity. There are no independent dynamical equations that tell us how the Newtonian field of gravity changes with time. When material bodies move around or change shape, their fields of gravity adapt instantaneously everywhere in space to reflect these changes.

In relativistic mechanics, on the other hand, fields take on a completely different meaning. The reason is that instantaneous action-at-a-distance cannot take place. If matter is moved, the current view is that it will take some time before the field of gravity adjusts to the new positions, because no signal can travel faster than light. As we understand it today, gravity is mediated by a field that emanates from massive bodies and in the manner of light takes time to travel through a distance. If the Sun were suddenly to blink out of existence, it would take eight long minutes before daylight was switched off and the Earth set free in space. Due to relativity, force fields must also travel independently, obey their own equations of motion, and carry physical properties such as energy and momentum. Electromagnetic waves bringing radio and TV signals to us are examples of force fields thus liberated from their origin. Gravitational waves have not yet been observed directly. Indirectly they have been observed in the spin-down of binary neutron star systems, which cannot be fully understood unless gravitational radiation is taken into account [WT05].

Even if we do not deal with relativistic theories of the continuum, and therefore may consider the gravitational field to be merely a mathematical convenience, it may nevertheless be wise, at least in the back of our minds, to think of the field of gravity as having an independent physical existence. Then we shall have no philosophical problem endowing it with physical properties such as energy, even in matter-free regions of space.

Is matter *really* discrete or continuous?

Although continuum physics is always an approximation to the underlying discrete molecular level, this is not the end of the story. At a deeper level it turns out that matter is best described by another continuum formalism, relativistic quantum field theory, in which the discrete particles—electrons, protons, neutrons, nuclei, atoms, and everything else—arise as quantum excitations of the fields. Relativistic quantum field theory without gravitation emerged in the middle of the twentieth century as *the* basic description of the subatomic world, but in spite of its enormous success it is still not clear how to include gravity.

Just as the continuity of macroscopic matter is an illusion, the quantum field continuum may itself one day be replaced by even more fundamental discrete or continuous descriptions of space, time, and matter. It is by no means evident that there could not be a fundamental length in nature setting an ultimate lower limit to distance and time, and theories of this kind have in fact been proposed [Whe89]. It appears that we do not know, and perhaps will never know, whether matter at its deepest level is truly continuous or truly discrete.

Problems

1.1 Consider a small volume V of a much larger volume of gas, such that the probability for any molecule to be found in V is exceedingly small. If the average number of molecules in V is known to be N , the probability of finding precisely n of the molecules in V is given by the Poisson distribution,

$$\Pr(n|N) = \frac{N^n}{n!} e^{-N}.$$

Show that

- (a) It is normalized.
- (b) The mean value is $\langle n \rangle = N$.
- (c) The variance is $\Delta N^2 = \langle (n - N)^2 \rangle = N$, that is $\Delta N = \sqrt{N}$.

1.2 The Lennard–Jones potential is often used to describe the interaction energy between two neutral atoms. It is given by the conventional formula

$$V(r) = 4\epsilon \left(\frac{\sigma^{12}}{r^{12}} - \frac{\sigma^6}{r^6} \right),$$

where r is the distance between centers of the atoms. The parameters ϵ and σ have dimensions of energy and length.

- (a) Determine the equilibrium distance $r = a$ where the potential is minimal, and its minimal value.
- (b) Determine the leading behavior of the potential around minimum.
- (c) Calculate the frequency of radial harmonic vibrations around equilibrium with one atom held fixed.
- (d) As an example, take argon, which has molar mass $M_{\text{mol}} = 40 \text{ g mol}^{-1}$, molar energy $\epsilon = 1 \text{ kJ mol}^{-1}$, and equilibrium distance $a = 2.87 \text{ \AA}$.

1.3 Consider a collection of N identical molecules (a “material particle”) taken from a large volume of gas. Let the instantaneous molecular velocities be \mathbf{v}_n for $n = 1, 2, \dots, N$. Collisions with other molecules in the gas at large will randomly change the velocity of each of the selected molecules, but if there is no overall drift in the gas, the velocity of individual molecules should average out to zero, $\langle \mathbf{v}_n \rangle = \mathbf{0}$, the velocities of different molecules should be uncorrelated, $\langle \mathbf{v}_n \mathbf{v}_m \rangle = \mathbf{0}$ for $n \neq m$, and the average of the square of the velocity should be the same for all molecules, $\langle \mathbf{v}_n^2 \rangle = v_0^2$.

(a) Show that the root-mean-square average of the center-of-mass velocity of the collection equals v_0/\sqrt{N} .

1.4 Any distance function must satisfy the axioms

$$\begin{aligned} d(\mathbf{a}, \mathbf{a}) &= 0, \\ d(\mathbf{a}, \mathbf{b}) &= d(\mathbf{b}, \mathbf{a}), && \text{(symmetry)} \\ d(\mathbf{a}, \mathbf{b}) &\leq d(\mathbf{a}, \mathbf{c}) + d(\mathbf{c}, \mathbf{b}). && \text{(triangle inequality)} \end{aligned}$$

Show that the Cartesian distance function (1.14) satisfies these axioms.