Gravity

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The force of gravity determines to a large extent the way we live. It is certainly the force about which we have the best intuitive understanding. We learn the hard way to rise against it as small children, to keep it at bay as adults, only to be brought down by it in the end.

Newton gave us the theory of gravity and the mathematics to deal with it. In a world where things only seem to get done by push and pull, man suddenly had to accept that the Earth could act on the distant Moon—and the Moon back on Earth. After Newton everybody had to suppress a feeling of horror for action at a distance and accept that gravity instantaneously could jump across the emptiness of space and tug at distant bodies. It took more than two centuries and the genius of Einstein to undo this learning. There is *no* action at a distance. As we understand it today, gravity is mediated by a field which emerges from massive bodies and in the manner of light takes time to travel through a distance. If the Sun were suddenly to blink out of existence, it would take eight long minutes before daylight was switched off and the Earth set free in space.

In this chapter we shall study the interplay between mass and the Newtonian field of gravity, and derive the equations governing this field and its interactions with matter. Some basic knowledge of gravity is assumed in advance, and the motivation behind this chapter is mainly to gain familiarity with the methods of field calculus in the comfortable environment of Newtonian gravity.

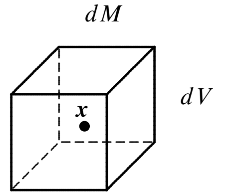
1 Mass density

In the continuum approximation the mass density is a field, $\rho(\boldsymbol{x})$, assumed to exist everywhere in space. If there is no mass in a region, the mass density is simply set to zero there. Knowing this field, we may calculate the mass of a material particle occupying a small volume dV near the point $\boldsymbol{x}$,

$$\boxed{dM = \rho(\boldsymbol{x})\, dV.} \tag{1}$$

Although not made explicit here, the density may depend on time. We shall also permit ourselves to suppress the space variable $\boldsymbol{x}$ and just write $dM = \rho\, dV$, whenever such notation is unambiguous.

Even if we shall usually think of the mass density field as varying smoothly throughout space, it is sometimes necessary to allow for discontinuous boundaries in material bodies. Often such discontinuities are "real" in the sense that the transition between different materials takes place on the molecular scale, for example between two bodies that touch each other. In the plot of the mass density of the Earth (Figure 1), the transition between core and mantle is for our purposes best described by a discontinuity, even if the actual transition zone is known to be quite broad [1].



The volume dV occupied by a material particle may take any shape, here cubical. The nominal position of the particle $\boldsymbol{x}$ is the center of mass.

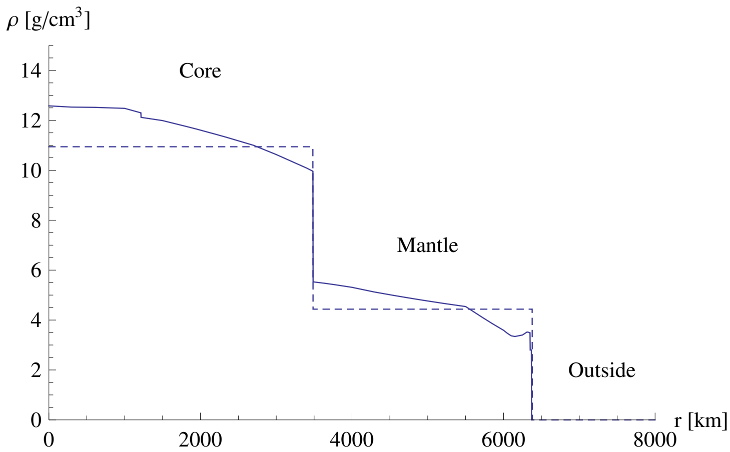
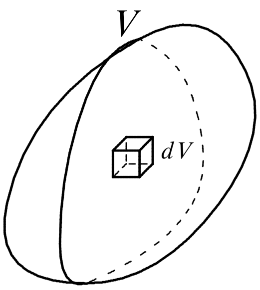


Figure 1. The mass density of the Earth as a function of distance r from the center with the surface at $r = 6371$ km (from the standard Earth model CM-[Lide 1996]). There is a sharp break, called the Gutenberg discontinuity, at the transition between the liquid iron core and the solid stone mantle at $r = 3485$ km. The broken lines indicate the average densities in the core (10.9 g cm^{-3}) and in the mantle (4.5 g cm^{-3}). The drop in density from core to mantle is in fact larger than the drop in density at the surface of the Earth.

Total mass of a body



The total mass in a volume is obtained by integrating ("summing") over all material particles in the volume. .

The mass density is a *local* quantity, a field defined in every point of space (and at every instant of time). In continuum physics the material contained in *any* volume may be viewed as a "body", and the total mass of a body with volume V is obtained by integrating the mass density over the volume,

$$M = \int_V dM = \int_V \rho(\boldsymbol{x})\, dV = \lim_{N\to\infty} \sum_{n=1}^{N} \rho(\boldsymbol{x}_n) dV_n. \tag{2}$$

As shown by the last expression, the integral should physically be understood in the Riemannian sense as the limit of a sum over a huge number N of tiny material particles, each having a volume dV_n near $\boldsymbol{x}_n$ and together filling out the volume V. The integral sign $\int$ is in fact a stylized version of the letter S (for "sum"). The shape of each individual volume dV_n and the precise position of its "anchor point" $\boldsymbol{x}_n$ is unimportant as long as the density is a smooth function (see the discussion in section 1.2). If the density depends on time, or if the volume changes shape and size with time, the total mass will also depend on time.

Center of mass

The *center of mass* of a body is naturally calculated by averaging the position $\boldsymbol{x}$ over the masses of all material particles,

$$\boldsymbol{x}_{\rm M} = \frac{1}{M}\int_V \boldsymbol{x}\, dM = \frac{1}{M}\int_V \boldsymbol{x}\, \rho(\boldsymbol{x})\, dV. \tag{3}$$

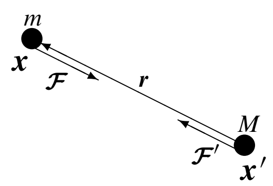
In the Newtonian mechanics of particles and stiff bodies, the center of mass of a body plays an important role, because it moves like a point particle under the influence of the total force acting on the body (see Appendix CM-A). This is in principle also true in continuum mechanics but is not nearly as useful because the shape of a body may change drastically over longer time-spans. Think for example of a bucket of oil thrown into a waterfall. It would be physically meaningless to speak about a well-defined "body of oil" and its center of mass at much later times.

2 Newton's law of gravity

In his *Principia* from 1687 (see figure CM-1.4) Isaac Newton concluded that the universal gravitational attraction between the Sun and the planets as well as between the Earth and the Moon had to act along the line between these objects and vary inversely with the square of the distance between them. The distances between the astronomical objects were so great that they could be considered to be point particles within the precision obtainable at that time. By comparing with the known strength of gravity at the surface of the Earth (the proverbial apple), Newton could predict the strength of gravity between all bodies.

Gravitational force between point particles

To formulate Newton's law in modern vector language, we place two point particles with masses m and M in the points $\boldsymbol{x}$ and $\boldsymbol{x}'$. We denote the position vector of m relative to M by $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$, and the corresponding unit vector by $\boldsymbol{e}_r = \boldsymbol{r}/r$ where $r = |\boldsymbol{r}| = |\boldsymbol{x} - \boldsymbol{x}'|$ is the distance between the particles. The gravitational force exerted on m by M is then in three different formulations,



Geometry and forces of gravity between two point particles.

$$\mathcal{F} = -G\frac{mM}{r^2}\boldsymbol{e}_r = -GmM\frac{\boldsymbol{r}}{r^3} = -GmM\frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3}, \tag{4}$$

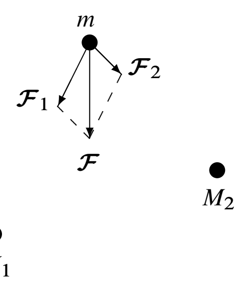
where G is the universal gravitational constant that sets the strength of gravity. The negative sign asserts that gravitation is always attractive. If one interchanges marked and unmarked quantities, the last expression shows clearly that the force that m exerts on M is of the same magnitude but opposite direction, $\mathcal{F}' = -\mathcal{F}$. This is in accordance with Newton's Third Law which states that for every action there is an equal and opposite reaction.

The gravitational constant is hard to determine with high precision. The recommended 2006 value, $G = 6.67428(67) \times 10^{-11}\ \mathrm{N\ m^2\ kg^{-2}}$, has an embarrassingly large uncertainty of one part in 10^4 CM-[1]. The inverse square law has been well tested at planetary distances during the last centuries, but only recently at the submillimeter scale (footnote **??**).

Electrostatics versus gravistatics: Electrostatic and "gravistatic" forces seem superficially alike in that they are both inversely proportional to the square of the distance, which gives them infinite range. They are in fact the only fundamental forces in nature with macroscopic infinite range. But where electric charge can be both positive and negative, mass is always positive, implying that there are no "neutral" bodies unaffected by gravity, nor bodies that are repelled by the gravity of other bodies, also called antigravity. It takes General Relativity CM-[Weinberg 1972] to see that gravity and electrostatics are very different at a deeper level.

Gravitational force on a point particle

In continuum physics bodies are certainly not pointlike but described by extended mass distributions representing myriads of material particles. A basic empiric property of gravity is that it is *additive*, so that the gravitational force on a point particle caused by a collection of point particles is the sum of the individual forces. For a body with mass distribution $\rho(\boldsymbol{x})$ in the volume V, we simply add the forces from all the material particles making up the body,



The gravitational force from two point particles, M_1 and M_2, on a third, m, is simply the sum, $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, of the forces from each of them.

$$\mathcal{F} = -Gm\int_V \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3}\rho(\boldsymbol{x}')dV', \tag{5}$$

where dV' denotes an infinitesimal volume element near $\boldsymbol{x}'$.

Relativistic non-additivity: The principle of gravitational additivity is, however, slightly compromised by the relativistic equivalence of mass and energy, $E = mc^2$. The molecules making up a solid or fluid body are attracted to each other by various van der Waals forces of electromagnetic origin, as discussed in section 1.1. This gives rise to a negative potential energy, besides which there is also the positive kinetic energy in the thermal vibrations. For solids and fluids the net binding energy is negative such that the mass of the body is slightly lower than the sum of the masses of its constituent molecules and thereby diminishes the gravitational force the body exerts on a point particle. For ordinary matter, for example water, the molecular mass defect is only of relative magnitude 10^{-12}. Gravitational attraction also binds matter, especially in astronomical bodies like planets and stars. For a body with mass one kilogram the relative gravitational mass defect is only about 10^{-27}, whereas for the Earth it is 10^{-9}, for the Sun 10^{-6}, and for a neutron star a few percent.

3 The field of gravity

Gravity is unique among all forces in nature in being proportional to the mass of the particle it acts upon. In Newton's Second Law for a point particle, $m\ddot{\boldsymbol{x}} = \boldsymbol{\mathcal{F}}$, the mass can simply be divided out, so that the equation of motion for any point particle in a gravitational field becomes the same,

$$\ddot{x} = \boldsymbol{g}(\boldsymbol{x}). \tag{6}$$

where the field $\boldsymbol{g}(\boldsymbol{x})$ is called the *field of gravity*, the *gravitational field*, or just *gravity*. The field of a point particle of mass M at $\boldsymbol{x}'$ is read off from (4),

$$\boldsymbol{g}(\boldsymbol{x}) = -G\frac{M}{r^2}\boldsymbol{e}_r = -GM\,\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}, \tag{7}$$

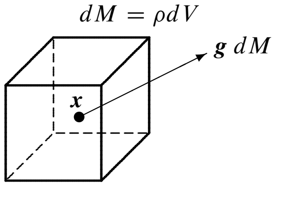
and that of an extended mass distribution from (5)

$$\boldsymbol{g}(\boldsymbol{x}) = -G\int_V \frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,\rho(\boldsymbol{x}')dV'. \tag{8}$$

The gravitational field is simply the force of gravity on a unit mass point particle.

The gravitational field imparts a common acceleration to all point particles and therefore also to the center-of-mass of all extended bodies (see appendix CM-A). Given the same initial conditions — i.e., the same initial position and velocity — all material bodies will follow the same orbits in a gravitational field. This confirms the law Galileo found empirically, that all bodies fall in the same way independent of their mass.

Curved space: The identical behavior of all bodies in the field of gravity allows one to look upon the gravitational field as a property of space and time, rather than simply a vehicle for gravitational interaction. As a consequence there is no way we can distinguish between gravitational forces and the inertial (so-called fictitious) forces experienced in accelerated motion. The indistinguishability of gravitional and motional acceleration was raised to a fundamental law, the *Principle of Equivalence*, by Einstein in his *General Theory of Relativity* from 1916, in which gravity is caused by the geometric curvature of space and time CM-[Weinberg 1972].



The force of gravity on a material particle of mass dM and volume dV is $d\boldsymbol{\mathcal{F}} = \boldsymbol{g}\,dM = \rho\,\boldsymbol{g}\,dV$.

A material particle is always embedded in an environment that exerts other forces than gravity on it. The part of the force due to gravity on a material particle of mass $dM = \rho dV$ at $\boldsymbol{x}$ is,

$$\boxed{d\boldsymbol{\mathcal{F}} = \boldsymbol{g}(\boldsymbol{x})\,dM = \rho(\boldsymbol{x})\,\boldsymbol{g}(\boldsymbol{x})dV,} \tag{9}$$

We shall again suppress space (and time) variables and write $d\boldsymbol{\mathcal{F}} = \rho\boldsymbol{g}\,dV$ when it is unambiguous.

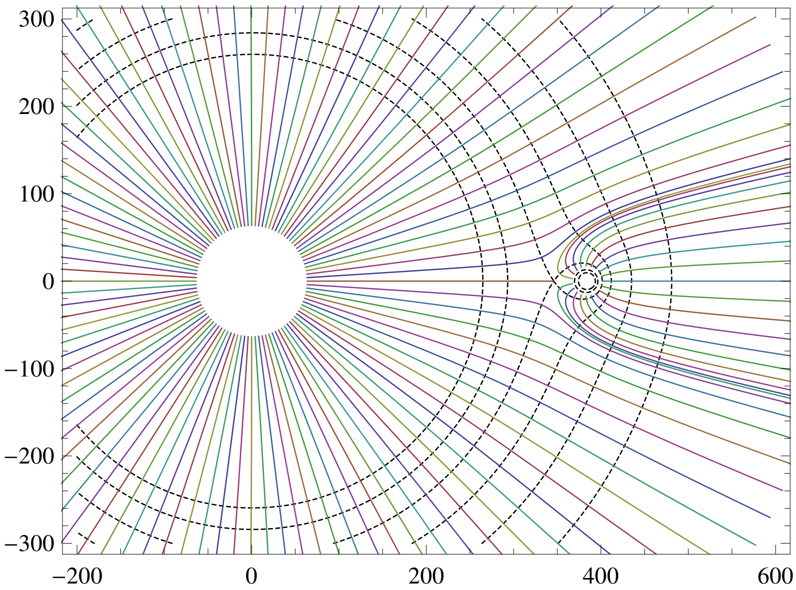


Figure 2. The gravitational field and a few nearly circular equipotential surfaces between Earth and Moon. You should imagine rotating this picture around the Earth–Moon axis. The drawing is to scale, except for two regions of 10 times the sizes of the Earth and the Moon that have been cut out for technical reasons. The field lines are plotted everywhere with a density proportional to the field strength. The numbers on the frame are coordinates centered on Earth in units of 1000 km. The Moon appears to have a streaming "mane of hair" because all the field lines ending on its surface have to come in from spatial infinity and cannot cross the lines of Earth's field. Note that the equipotential surfaces cross in the unique point where the gravitational fields of the Earth and Moon cancel each other.

Total force and moment of force

The total gravitational force on a body of volume V, also called the *weight* of the body, is obtained by adding the weights of every material particle in the body,

$$\mathcal{F} = \int_V \boldsymbol{g}\, dM = \int_V \rho\, \boldsymbol{g}\, dV. \tag{10}$$

The total gravitational force is a vector which (together with other forces) determines how the body as a whole moves.

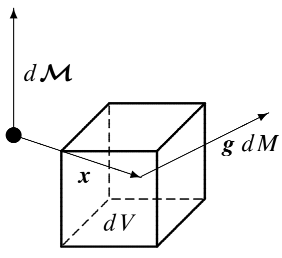
The total gravitational *moment of force*,

$$\mathcal{M} = \int_V \boldsymbol{x} \times \boldsymbol{g}\, dM = \int_V \boldsymbol{x} \times \rho\, \boldsymbol{g}\, dV. \tag{11}$$

is a vector which (together with moments of other forces) determines how a body as a whole rotates around the origin. The total moment depends on the point around which we choose to calculate it, here the origin. In the terminology of section B.6 the moment is an improper axial vector. If we instead calculate the moment around another point than the origin of the coordinate system, say $\boldsymbol{x} = \boldsymbol{c}$, it becomes

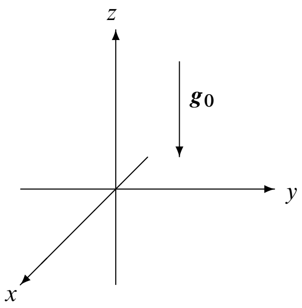
$$\mathcal{M}(\boldsymbol{c}) = \int_V (\boldsymbol{x} - \boldsymbol{c}) \times \rho \boldsymbol{g}\, dV = \mathcal{M} - \boldsymbol{c} \times \mathcal{F}. \tag{12}$$

This shows that *the total moment is independent of the choice of origin of the coordinate system if (and only if) the total force vanishes*.



The moment of gravity for a material particle with mass $dM = \rho\, dV$ is $d\mathcal{M} = \boldsymbol{x} \times \boldsymbol{g}\, dM$.

Constant gravity



The flat-earth coordinate system.

At the surface of the Earth, it is often convenient to employ a "flat-earth" coordinate system with vertical z-axis and a constant gravitational field of the form $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}_0 = (0, 0, -g_0)$ where g_0 is a positive constant. The flat-earth approximation can of course only be meaningful for regions that are so small that the deviation from flatness due to Earth's spherical shape is insignificant. At the surface of Earth, the magnitude of gravity is roughly equal to standard gravity, defined by convention to be exactly 9.80665 m s^{-2} with no uncertainty CM-[1].

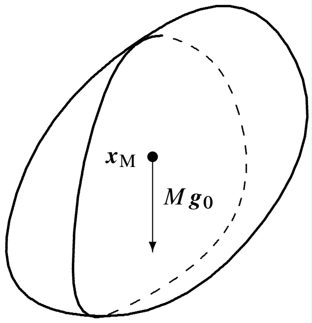
Variations in gravity: The actual local gravitational acceleration at the surface of the Earth depends on many factors, for example latitude, nearby mass concentrations, and the positions of the Moon and Sun. The variations in the local gravitational acceleration has been determined with a precision of 3×10^{-9} in an experiment using atom interferometry CM-[PCC99]. Galileo's law was verified in the same experiment to within 7×10^{-9} by comparing the measured values of the gravitational acceleration for a macroscopic body and for a cesium atom, in effect a modern version of Galilei's famous "leaning tower of Pisa" experiment.

In constant gravity, the weight of a body (10) becomes the familiar

$$\mathcal{F} = \int_V \rho\, \boldsymbol{g}_0\, dV = \left(\int_V \rho\, dV\right) \boldsymbol{g}_0 = M \boldsymbol{g}_0, \tag{13}$$

where M is the total mass (2). The moment of gravity becomes,

$$\mathcal{M} = \int_V \boldsymbol{x} \times \rho \boldsymbol{g}_0\, dV = \left(\int_V \rho\, \boldsymbol{x}\, dV\right) \times \boldsymbol{g}_0 = \boldsymbol{x}_M \times M \boldsymbol{g}_0. \tag{14}$$



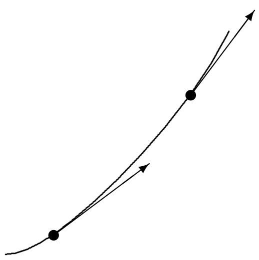
In a constant gravitational field $\boldsymbol{g}_0$, the weight of a body may be viewed as concentrated at the position of the center of mass, $\boldsymbol{x}_M$.

where $\boldsymbol{x}_M$ is the center of mass (3). This shows that in constant gravity, the total force of gravity as well as its moment is the same as that of a point particle with mass equal to the total mass of the body, situated at the center of mass. Evidently, the moment of gravity in a constant field vanishes if calculated with respect to the center of mass.

Visualizing gravity

A visual impression of the gravitational field may be given by a picture of the *field lines*, defined to be families of curves that at a given instant t_0 have the gravitational field as tangent (see figure 2). This means that the curves are solutions to the first-order vector differential equation

$$\frac{d\boldsymbol{x}}{ds} = \boldsymbol{g}(\boldsymbol{x}, t_0), \tag{15}$$


Field lines are everywhere tangent to the instantaneous field.

where s is a running parameter along the curve. This parameter is not the time, but has dimension of time squared because $\boldsymbol{g}$ has dimension of length per unit of time squared. The solutions are of the form $\boldsymbol{x} = \boldsymbol{x}(s, \boldsymbol{x}_0, t_0)$ with $\boldsymbol{x}_0$ being the starting point at $s = 0$. The field lines form an instantaneous picture of the field at time t_0, and cannot be directly related to particle orbits. A planet may, for example, move in a nearly circular orbit which is everywhere orthogonal to the field lines.

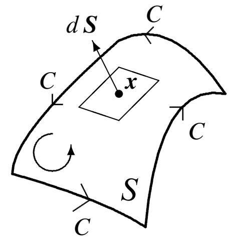
Field lines have the very important property that they can never cross. For if two field lines crossed in a point $\boldsymbol{x}$, then by (15) there would have to be two different values of the gravitational field at the same point, and that is impossible (except when the field vanishes, as it does in one point of figure 2). As will be shown below all gravitational field lines have to come in from infinity and end on masses, and we shall also see that field lines do not form closed loops.

4 Gravitational flux

Gravitational flux is a measure of how much gravity "streams" through an oriented surface S of any shape. It is defined by the surface integral

$$\int_S \boldsymbol{g}(\boldsymbol{x}) \cdot d\boldsymbol{S} = \lim_{N\to\infty} \sum_{n=1}^{N} \boldsymbol{g}(\boldsymbol{x}_n) \cdot d\boldsymbol{S}_n. \tag{16}$$

As previously discussed on page CM-614, the integral should be understood as the limit of a huge sum over tiny planar patches of the surface, called *surface elements*, each being represented by an area vector $d\boldsymbol{S}$ which is orthogonal to the surface and has length $dS = |d\boldsymbol{S}|$ equal to the area of the patch. There is nothing intrinsic in the surface which tells us the direction of the surface element (in the language of section CM-B.6 we would say that the surface element is an axial vector). Having made a choice, all surface elements are required to be oriented to the same side of the surface.

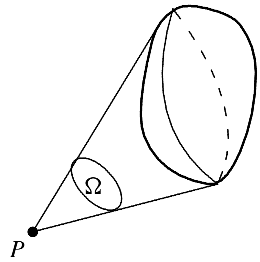


All normals to an oriented open surface must have a consistent orientation, meaning that they point to the same side of the surface. The perimeter curve C must be oriented in the same way as the surface, here by means of the right-hand rule.

Flux and solid angle

There is an interesting and important relation between the gravitational flux from a point mass through a surface, and the solid angle subtended by the surface as observed from the position of the point mass. The *solid angle* subtended by any object seen from a given observation point is defined as the area that the object projects on the surface of a unit sphere centered at this point. According to this definition, the total solid angle of any convex object observed from anywhere inside its volume is equal to the area 4π of the unit sphere.

The infinitesimal solid angle $d\Omega$ subtended by a surface element $d\boldsymbol{S}$ near $\boldsymbol{x}$ as seen from the point $\boldsymbol{x}'$, is obtained in two steps. First the area of the surface element orthogonal to the line-of-sight is calculated by projecting the area vector $d\boldsymbol{S}$ on the direction $\boldsymbol{e}_r = \boldsymbol{r}/r$ of the relative position vector $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$. Next the projected area is scaled to the unit sphere by dividing with r^2,



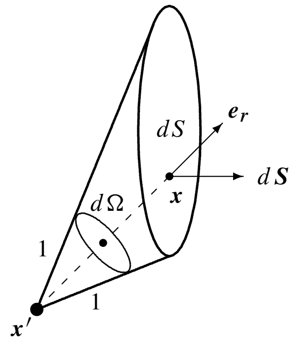
The solid angle Ω subtended by the ball of ice in an ice-cream cone seen from its tip P equals the area of its "shadow" on the unit sphere centered at P.

$$d\Omega = \frac{\boldsymbol{e}_r \cdot d\boldsymbol{S}}{r^2}. \tag{17}$$

Surprisingly, this is of the same functional form as the gravitational field (7) of a point particle, allowing us elegantly to express the contribution to the gravitational flux from the surface element as,

$$\boldsymbol{g} \cdot d\boldsymbol{S} = -GM\, d\Omega.$$

This result is quite general and is valid for any position $\boldsymbol{x}'$ of the point mass, as long as the solid angle is calculated from that position. Notice that the solid angle $d\Omega$ can be negative if the observation point lies in front of the surface element rather than behind it. In the margin figure the solid angle is actually positive.

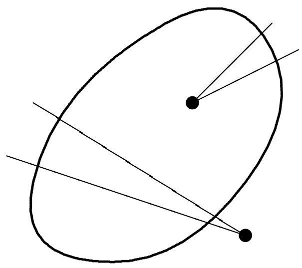


The solid angle subtended by a surface element near $\boldsymbol{x}$ seen from the point $\boldsymbol{x}'$ equals the projection of the surface element on the direction $\boldsymbol{e}_r$ of the relative position vector, divided by the square of its length $r = |\boldsymbol{x} - \boldsymbol{x}'|$.

Integrating over the surface S we obtain

$$\boxed{\int_S \boldsymbol{g} \cdot d\boldsymbol{S} = -GM\Omega} \tag{18}$$

where Ω is the solid angle that S subtends when observed from the position of the point mass. If the surface is convoluted, the line-of-sight from the point mass to a surface element may cross the surface more than once. Each time it crosses, the sign of the contribution to the solid angle will change, such that it cancels the preceding contribution. We shall now see how this works out for a closed surface.



The lines of sight from a point inside a convex surface crosses the surface once, whereas they cross twice if the mass is outside.

Gauss' law

Consider now a *closed convex surface* containing the point mass M somewhere in the enclosed volume. All the little solid angles add up to 4π because the line-of-sight from the particle in any direction crosses the convex surface exactly once. If, on the other hand, the surface does not contain the point mass, the line-of-sight from the particle will always cross the surface twice. The two contributions to the flux will then cancel because the solid angles have the same magnitudes but opposite signs (since all normals are directed out of the volume). In other words, for any point mass M with position $\boldsymbol{x}'$ we have,

$$\oint_S \boldsymbol{g} \cdot d\boldsymbol{S} = \begin{cases} -4\pi GM & \text{for } \boldsymbol{x}' \in V \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

The circle through the surface integral on the left merely signals that the surface is closed. This result is, in fact, valid for any orientable closed surface, convex or not. For a convoluted surface, the line-of-sight from the inside will always cross the surface an odd number of times, and all the contributions along the line-of-sight cancel each other, except for one. If the particle is outside the volume the line-of-sight will cross an even number of times and all contributions cancel. The conclusion is that the above equation holds in full generality.

Finally, adding together the contributions from all the material particles in the volume V enclosed by the surface S, we get the *global* form of *Gauss' law*,

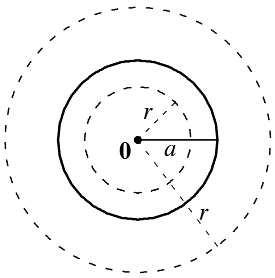
$$\boxed{\oint_S \boldsymbol{g} \cdot d\boldsymbol{S} = -4\pi G \int_V \rho \, dV,} \tag{20}$$

where the integral on the right hand side is the total mass contained in the volume V.

Gauss' theorem CM-(C.15) allows us to convert the integral on the left hand side into a volume integral, such that $\int_V \nabla \cdot \boldsymbol{g} \, dV = -4\pi G \int_V \rho \, dV$ for all volumes V. Letting the volume shrink down to nothing around the point $\boldsymbol{x}$, it follows that the integrands must be equal for all $\boldsymbol{x}$, and we obtain the *local* form of Gauss' law,

$$\boxed{\nabla \cdot \boldsymbol{g} = -4\pi G\rho.} \tag{21}$$

It is a partial differential equation for $\boldsymbol{g}$ without reference to any surface or volume, and is one of the two fundamental field equations for gravity. We shall return to these equations in section 6.



Spherical planet with radius a. Gauss' law can be used to calculate the field on any sphere inside or outside the planet (dashed).

Field of a spherical body

Except for spherically symmetric bodies, like planets and stars, the global form is not very useful in practice. The mass distribution $\rho(r)$ of a spherically symmetric body is only a function of the distance $r = |\boldsymbol{x}|$ from its center, which is here chosen to be at the origin of the coordinate system. Because of the spherical symmetry, the gravitational field must everywhere be directed radially away from the center,

$$\boldsymbol{g}(\boldsymbol{x}) = g(r)\,\boldsymbol{e}_r, \tag{22}$$

where $\boldsymbol{e}_r = \boldsymbol{x}/r$ is the radial unit vector and $g(r)$ is a scalar function of r.

Let us choose the volume in the form of a concentric sphere with radius r. The right-hand side of Gauss' global law then becomes $-4\pi GM(r)$ where

$$M(r) = \int_{s \le r} \rho(s) \, dV = \int_0^r \rho(s) 4\pi s^2 \, ds. \tag{23}$$

is total mass inside the sphere.

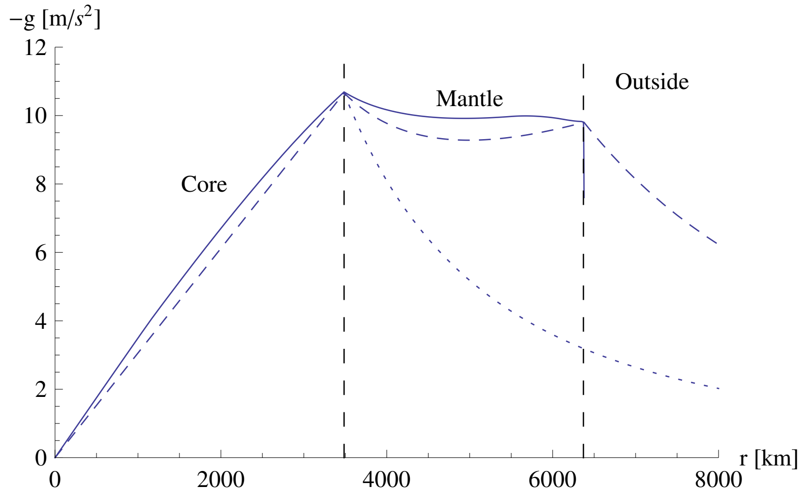


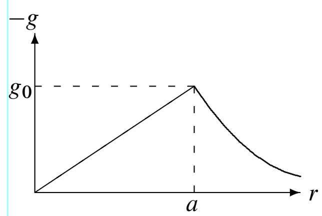
Figure 3. The strength of gravity $-g(r)$ in the core and mantle as well as outside the Earth as a function of distance from the center. The solid curve is obtained by numerical integration over the density data used in figure 1. The strength of gravity grows roughly linearly from the center to the core/mantle boundary at $r = 3485$ km, and decreases slightly in the mantle due to the sharp drop in mass density at the boundary. The dotted dropping line is the core field itself. The dashed lines are obtained from the two-layer model of the Earth (problem 5).

The surface elements on the sphere are everywhere pointing radially outwards, $d\boldsymbol{S} = \boldsymbol{e}_r\, dS$, so that $\boldsymbol{g}(\boldsymbol{x}) \cdot d\boldsymbol{S} = g(r)dS$. Since $g(r)$ is constant on the spherical surface, it goes outside the integral on the left-hand side of Gauss' law which becomes $4\pi r^2 g(r)$, so that

$$\boxed{g(r) = -G\frac{M(r)}{r^2}.} \tag{24}$$

In figure 3 the value of $-g(r)$ is plotted for the Earth as a function of the radial distance. One notes the surprising fact that the strength of gravity is actually larger (by about 9%) at the core-mantle boundary than on the surface. This is caused by the heavy iron core of the earth with a mass density nearly three times larger than in the mantle.

It follows from (24) that in the vacuum *outside* a spherical mass distribution where $M(r)$ equals the total mass, the field is *exactly* the same as that of a point particle at the center with mass equal to the total mass of the body. Although as we shall see below the field at great distances from an arbitrary body is always approximately that of a point particle, we learn that the field is of this form everywhere around a perfectly spherical body. There are *no near-field corrections to the gravitational field of a spherical body*. Without this wonderful property, Newton could never have related the strength of gravity at the surface of the Earth—iconized by the fall of an apple—to the strength of gravity in the Moon's orbit.



The strength of gravity for a planet with constant density as a function of distance from the center. Gravity is maximal at the surface.

Example 1 [Spherical planet with constant density]: For a spherical planet with radius a and density function

$$\rho(r) = \begin{cases} \rho_0 & \text{for } r < a, \\ 0 & \text{for } r > a, \end{cases} \tag{25}$$

the mass function becomes

$$M(r) = \frac{4\pi}{3}\rho_0 \begin{cases} r^3 & \text{for } r < a, \\ a^3 & \text{for } r > a. \end{cases} \tag{26}$$

The strength of gravity is then according to (24) found to be,

$$g(r) = -\frac{4\pi}{3} G\rho_0 \begin{cases} r & \text{for } r < a, \\ a^3/r^2 & \text{for } r > a. \end{cases} \tag{27}$$

Inside the planet, the field rises linearly with the distance from the center. In figure 3 this is seen to be quite well fulfilled for the core of the Earth, but certainly not for the mantle.

5 Gravitational potential

Although the field of gravity is a vector field with three components, there is really only one functional degree of freedom underlying the field, namely the mass density field. The relationship between gravity and density, expressed through the integral (8), is *non-local*. This means that the gravitational field in a point $\boldsymbol{x}$ depends on the mass density in points $\boldsymbol{x}'$ that in principle may be arbitrarily far away.

Even if one cannot get rid of the non-locality, one can avoid having a three-to-one relationship by defining a new scalar field $\Phi(\boldsymbol{x})$, called the *gravitational potential*, also attributed to Gauss (1840). The potential is, as we shall see below, also non-locally related to the mass density, but the gravitational field itself can be calculated locally as minus the gradient (C.2) of the potential,

$$\boxed{\boldsymbol{g}(\boldsymbol{x}) = -\boldsymbol{\nabla}\Phi(\boldsymbol{x}) = -\left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}\right).} \tag{28}$$

Due to the differentiation, the potential is only defined up to addition of an arbitrary constant. For a bounded mass distribution, this constant is normally fixed by requiring the potential to vanish at spatial infinity. We shall see below that the potential $\Phi(\boldsymbol{x})$ is in fact the potential energy of a unit mass particle in the point $\boldsymbol{x}$.

Potential of a point mass

To find an expression for the potential of a point mass, we first calculate the gradient of the distance $r = |\boldsymbol{x} - \boldsymbol{x}'|$ with $\boldsymbol{x}'$ held fixed,

$$\boldsymbol{\nabla} r \equiv \frac{\partial r}{\partial \boldsymbol{x}} = \frac{1}{2r}\frac{\partial\left(r^2\right)}{\partial \boldsymbol{x}} = \frac{1}{2r}\frac{\partial\left((\boldsymbol{x}-\boldsymbol{x}')^2\right)}{\partial \boldsymbol{x}} = \frac{\boldsymbol{x}-\boldsymbol{x}'}{r} = \boldsymbol{e}_r. \tag{29}$$

The gradient of the radial distance equals the unit vector in the radial direction.

The gradient of $1/r$ can now be calculated using the chain rule,

$$\boldsymbol{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\boldsymbol{\nabla} r = -\frac{\boldsymbol{e}_r}{r^2} = -\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}.$$

Comparing with the field of a point mass (7), we conclude that we can write $\boldsymbol{g}$ in the form (28) with

$$\Phi(\boldsymbol{x}) = -G\frac{M}{r} = -G\frac{M}{|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{30}$$

This is obviously the gravitational potential of a point mass M situated in $\boldsymbol{x}'$.

Potential of a mass distribution

By appealing to the additivity of gravity or by direct comparison with the field (8) from an extended body, the potential of a bounded mass distribution (a finite body) becomes,

$$\boxed{\Phi(\boldsymbol{x}) = -G\int_V \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}dV'.} \tag{31}$$

Since the mass density is always positive, the gravitational potential is always negative, provided (as here) the arbitrary constant is chosen such that the potential vanishes at infinite spatial distance.

The gravitational potential may be visualized by means of surfaces of constant potential, $\Phi(\boldsymbol{x}) = \text{const}$, called *equipotential surfaces*. The field of gravity, and thus the field lines, are always orthogonal to the equipotential surfaces, and if they are plotted with constant potential difference, the strength of the field will be inversely proportional to the distances between them. A few equipotential surfaces have been shown in the Earth–Moon plot in figure 2.

Asymptotic behavior

At large distances from a finite body the potential becomes that of a point particle situated at the center of mass $\boldsymbol{x}_M$ with mass M equal to the total mass of the body. To show this we put $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}_M$ and $\boldsymbol{r}' = \boldsymbol{x}' - \boldsymbol{x}_M$, so that $\boldsymbol{x} - \boldsymbol{x}' = \boldsymbol{r} - \boldsymbol{r}'$. If a is the maximal distance of any material particle in the body from the center of mass, we always have $r' \leq a$, and we can for $r \gg a$ to lowest order replace $|\boldsymbol{x} - \boldsymbol{x}'| = |\boldsymbol{r} - \boldsymbol{r}'|$ by $r = |\boldsymbol{r}|$ in the denominator of (31). Taking the factor $1/r$ outside the integral we find as promised,

$$\Phi(\boldsymbol{x}) \approx -G\frac{M}{r} \qquad \text{for } r \gg a, \tag{32}$$

where M is the total mass. A similar result follows for the gravitational field $\boldsymbol{g}$ by calculating the gradient of the asymptotic potential. The reason for choosing the center of mass, and not just an arbitrary point in the vicinity of the body, is that the corrections to the potential will then be of relative order $(a/r)^2$ rather than a/r (see problem 14).

Potential of spherical mass distribution

The potential of a spherical mass distribution can, like the density, only depend on the radial distance r. Using the chain rule and that $\boldsymbol{\nabla} r = \boldsymbol{e}_r$, we get

$$\boldsymbol{g}(\boldsymbol{x}) = -\boldsymbol{\nabla}\Phi(r) = -\frac{d\Phi(r)}{dr}\boldsymbol{\nabla} r = -\frac{d\Phi(r)}{dr}\boldsymbol{e}_r,$$

and by comparison with (22) we obtain

$$g(r) = -\frac{d\Phi(r)}{dr}. \tag{33}$$

Conversely, $\Phi(r)$ can be obtained by integrating $g(r)$, requiring the potential to vanish at infinity,

$$\Phi(r) = \int_r^\infty g(r)\,dr \tag{34}$$

The potential of the Earth is plotted in figure 4.

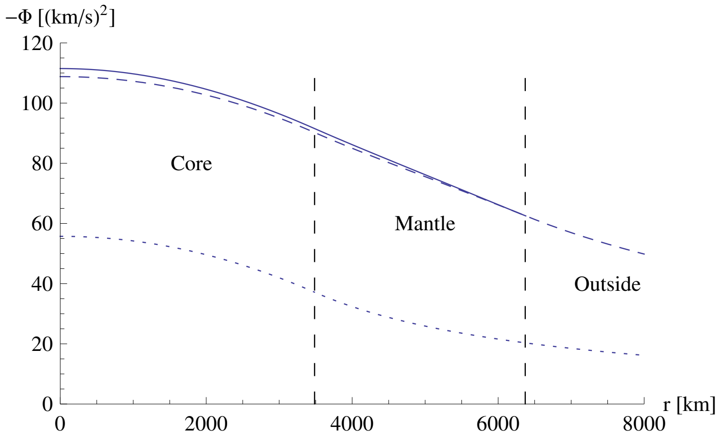


Figure 4. The gravitational potential of the Earth $-\Phi(r)$ as a function of distance from the center. The fully drawn curve is obtained by numerically integrating the field of gravity in figure 3. The dotted curve is the potential of the core alone, and the dashed curve is obtained from the two-layer model (problem 5). The vertical dashed lines indicate the positions of the sharp transitions in the mass density (see figure 1), which have been smoothed out here by the two integrations leading from the mass density to the potential.

Example 2 [Planet with constant mass density]: For a planet with constant mass density we obtain from the above expression and (27),

$$\Phi(r) = -\frac{2}{3}\pi G\rho_0 \begin{cases} 3a^2 - r^2 & r < a \\ 2\dfrac{a^3}{r} & r > a\,. \end{cases} \tag{35}$$

One may avoid integrating and instead verify that the derivative $-d\Phi/dr$ is indeed identical to (27) and that the the constant $3a^2$ has been determined such that the potential is continuous at the surface $r = a$.

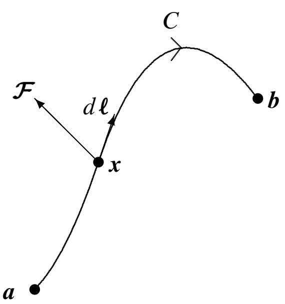
Gravitational work

According to the laws of Newtonian particle mechanics, the work performed by a force $\boldsymbol{\mathcal{F}}$ acting on a particle that is moved along an infinitesimal straight line from $\boldsymbol{x}$ to $\boldsymbol{x} + d\boldsymbol{\ell}$ is,

$$\boxed{dW = \boldsymbol{\mathcal{F}} \cdot d\boldsymbol{\ell}.} \tag{36}$$

The total work performed by the force when the particle is moved from $\boldsymbol{a}$ to $\boldsymbol{b}$ along the oriented path C then becomes a *curve integral*,

$$W = \int_C \boldsymbol{\mathcal{F}} \cdot d\boldsymbol{\ell} = \lim_{N\to\infty} \sum_{n=1}^{N} \boldsymbol{\mathcal{F}}(\boldsymbol{x}_n) \cdot d\boldsymbol{\ell}_n. \tag{37}$$



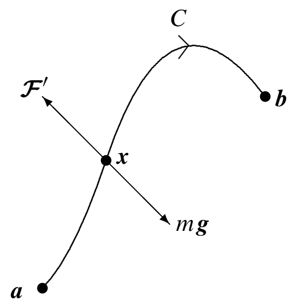
The oriented path C running from $\boldsymbol{a}$ to $\boldsymbol{b}$ can be viewed as a sequence of straight curve elements $d\boldsymbol{\ell}$.

As shown on the right, the integral should be understood as the limit of the sum over a huge number of small straight *curve elements* $d\boldsymbol{\ell}_n$ near the points $\boldsymbol{x}_n$.

As in real life, it is important to make completely clear who works for whom. When a particle falls freely in a gravitational field, it is the force of gravity that performs work on the particle while the particle follows the path of its natural motion and gains kinetic energy. If we want the particle to follow any other path, we must so to speak "by hand" cancel the force of gravity with an equal and opposite force, and with the tiniest extra force slowly guide the particle along the desired path. Suppose we move a point particle of mass m along a particular path C from position $\boldsymbol{a}$ to $\boldsymbol{b}$ in a static field of gravity $\boldsymbol{g}(\boldsymbol{x})$ where the gravitational force is $\mathcal{F} = m\boldsymbol{g}(\boldsymbol{x})$. To keep the particle on this path, *we* must provide an *external* force $\mathcal{F}' = -m\boldsymbol{g}(\boldsymbol{x})$ everywhere along the path to counter the force of gravity $m\boldsymbol{g}(\boldsymbol{x})$. Thus the work *we* must perform on the particle to move it along any desired path C is

$$W = \int_C \mathcal{F}' \cdot d\boldsymbol{\ell} = -m \int_C \boldsymbol{g} \cdot d\boldsymbol{\ell}. \tag{38}$$

The additional force that we need to guide the particle along the path is supposed to be so small that its contribution to our work can be ignored. If there is resistance against the particle's motion, as there is when moving a body through a viscous fluid, this has to be taken into account, but more about that later.



The force of gravity $m\boldsymbol{g}$ must be canceled by an external force $\mathcal{F}' = -m\boldsymbol{g}$ in order to move the particle slowly along any desired path between the end points $\boldsymbol{a}$ and $\boldsymbol{b}$.

Potential energy

The differential change in the potential along an infinitesimal curve element becomes according to Equation CM-(C.3)

$$d\Phi(\boldsymbol{x}) \equiv \Phi(\boldsymbol{x} + d\boldsymbol{\ell}) - \Phi(\boldsymbol{x}) = d\boldsymbol{\ell} \cdot \nabla\Phi(\boldsymbol{x}) = -\boldsymbol{g} \cdot d\boldsymbol{\ell}, \tag{39}$$

so the work we perform in moving the particle from $\boldsymbol{a}$ to $\boldsymbol{b}$ is obtained by adding differentials,

$$W = -m \int_C \boldsymbol{g} \cdot d\boldsymbol{\ell} = m \int_C d\Phi(\boldsymbol{x}) = m\,\Phi(\boldsymbol{b}) - m\,\Phi(\boldsymbol{a}). \tag{40}$$

The fact that the work is independent of the actual path of the particle shows that gravitational forces are *conservative*, and allows us to interpret $m\Phi(\boldsymbol{x})$ as the *potential energy* of a particle of mass m in the field of gravity. Therefore, the potential $\boldsymbol{\Phi}(\boldsymbol{x})$ itself is the potential energy of a unit mass particle.

Example 3 [Escape velocity]: The total energy of a point particle at $\boldsymbol{x}$ with velocity $\boldsymbol{v}$ is the sum of its kinetic energy and its potential energy,

$$E = \frac{1}{2} m\boldsymbol{v}^2 + m\Phi(\boldsymbol{x}) \tag{41}$$

From elementary mechanics we know that the total energy is conserved, that is, constant in time. Imagine now that a particle situated at the point $\boldsymbol{x}$ is ejected like a ball from a cannon with velocity $\boldsymbol{v}$, and arrives at spatial infinity with velocity $\boldsymbol{v}_\infty$. Conservation of energy tells us that

$$\tfrac{1}{2} m\boldsymbol{v}^2 + m\Phi(\boldsymbol{x}) = \tfrac{1}{2} m\boldsymbol{v}_\infty^2 \tag{42}$$

because $\Phi(\infty) = 0$. Since $\Phi(\boldsymbol{x}) < 0$, it follows that the smallest possible start velocity is obtained by taking $\boldsymbol{v}_\infty = \boldsymbol{0}$, so that the particle will escape if its velocity is equal to or larger than,

$$v_{\text{esc}} = \sqrt{-2\Phi}, \tag{43}$$

also called the *escape velocity*. Starting with precisely the escape velocity, the particle will arrive with zero velocity at infinity. Knowing the potential at a point is evidently equivalent to knowing the escape velocity from that point.

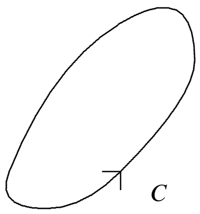
A particularly interesting case occurs when the potential becomes so deep that the escape velocity equals or surpasses the velocity of light c. In that case the body has turned into a black hole. Using the potential of a point mass (30) we find that this happens when the radius a of a spherical mass distribution satisfies

$$a < \frac{2GM}{c^2}. \tag{44}$$

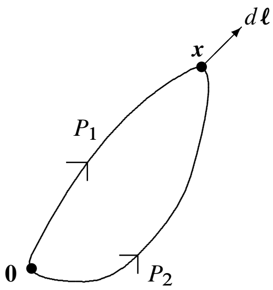
Being a non-relativistic calculation this is of course highly suspect, but accidentally it is exactly the same as the correct condition obtained in general relativity, where the right-hand side is called the Schwarzchild radius. For the Earth the Schwarzchild radius is about a centimeter, and for the Sun three kilometers.

Place	km s^{-1}
Earth surface	11.2
Mars surface	5.0
Moon surface	2.4
Sun surface	617.6
Earth orbit	42.1
Moon orbit	1.4
Neutron star	1×10^5
Black hole	3×10^5

Escape velocities from some places in the solar system, and a couple of exotic ones. Note that escaping from the orbit of Earth means escaping from the solar system whereas escaping from the orbit of the Moon only gets you free of Earth's gravity. The neutron star is assumed to have solar mass.



A gravitational field line cannot form a closed path C.



A closed curve $C = P_1 - P_2$ can be viewed as the difference between two paths connecting the same points. The curve element $d\boldsymbol{\ell}$ is a small prolongation beyond the end point of any path running from $\mathbf{0}$ to $\boldsymbol{x}$.

No closed loops of gravity

The work performed in moving a particle around a closed curve C must necessarily vanish because the curve begins and ends in the same point, $\boldsymbol{a} = \boldsymbol{b}$. The curve integral of the field of gravity, called its *circulation*, around any closed curve C must therefore vanish,

$$\oint_C \boldsymbol{g} \cdot d\boldsymbol{\ell} = 0. \tag{45}$$

This result tells us that *field lines cannot form closed loops*, because a field line is defined to be everywhere tangential to the field, $d\boldsymbol{\ell} \propto \boldsymbol{g}$, implying that the product $\boldsymbol{g} \cdot d\boldsymbol{\ell} \propto \boldsymbol{g}^2$ will always be positive. If a closed gravitational field line could exist, its circulation would always be positive, but that is impossible.

Conversely, *if the curve integral of a field* $\boldsymbol{g}$ *around any closed curve vanishes, the field must be a gradient field.* To demonstrate this consider the curve integral along some path $P(\boldsymbol{x})$ running from a fixed point, say the origin $\mathbf{0}$, to an arbitrary point $\boldsymbol{x}$, and define

$$\Phi_P(\boldsymbol{x}) = -\int_{P(\boldsymbol{x})} \boldsymbol{g}(\boldsymbol{x}') \cdot d\boldsymbol{\ell}'. \tag{46}$$

It now follows that this function depends only on the end point $\boldsymbol{x}$ and not on the path.

Two different paths, P_1 and P_2, connecting the same points form a closed curve $C = P_1 - P_2$, and since the circulation is known to vanish, the two paths must yield the same result, $\Phi_{P_1}(\boldsymbol{x}) = \Phi_{P_2}(\boldsymbol{x})$, when taking the orientations into account. One may consequently leave out the path P in $\Phi_P(\boldsymbol{x})$ and just call the function $\Phi(\boldsymbol{x})$. Finally, we must show that $\Phi(\boldsymbol{x})$ indeed has the gradient $-\boldsymbol{g}$, but that is easy. If we prolong any path from $\mathbf{0}$ to $\boldsymbol{x}$ by an arbitrary infinitesimal curve element $d\boldsymbol{\ell}$ we get the change in the potential $d\Phi(\boldsymbol{x}) \equiv \Phi(\boldsymbol{x} + d\boldsymbol{\ell}) - \Phi(\boldsymbol{x}) = -\boldsymbol{g}(\boldsymbol{x}) \cdot d\boldsymbol{\ell}$. But then Equation CM-(C.3) tells us that $\nabla\Phi(\boldsymbol{x}) \cdot d\boldsymbol{\ell} = -\boldsymbol{g} \cdot d\boldsymbol{\ell}$ for any $d\boldsymbol{\ell}$, and that is only possible if $\nabla\Phi = -\boldsymbol{g}$.

6 Field equations for gravity

So far we have established two local equations for the gravitational field. The first is the local version of Gauss' law (21), and the second the expression (28) for the gravitational field as the gradient of the potential. Since the curl CM-(C.5) applied to a gradient always vanishes, we have obtained the following two partial differential equations for the gravitational field alone,

$$\nabla \cdot \boldsymbol{g} = -4\pi G\rho, \qquad \nabla \times \boldsymbol{g} = \mathbf{0}. \tag{47}$$

Given the density ρ and suitable boundary conditions, these field equations in fact determine the gravitational field $\boldsymbol{g}$, although it takes a bit of mathematical work that we shall carry through below. With this result, we are thus liberated from dealing directly with the cumbersome integral in (8) and can employ the existing comprehensive mathematical toolbox for solving partial differential equations.

Poisson's equation

It follows from Stokes' theorem CM-(C.14) that $\nabla \times \boldsymbol{g} = \mathbf{0}$ implies the vanishing of the circulation (45) and thus guarantees the existence of the potential. Inserting $\boldsymbol{g} = -\nabla\Phi$ into the divergence equation (47), we obtain *Poisson's equation*[1],

$$\boxed{\nabla^2\Phi = 4\pi G\rho.} \tag{48}$$

[1]The field equations for electrostatics in vacuum are of exactly the same form with the mass density ρ replaced by the electric charge density ρ_e, the field of gravity $\boldsymbol{g}$ replaced by the electric field $\boldsymbol{E}$ and $-4\pi G$ replaced by $1/\epsilon_0$.

where

$$\nabla^2 = \nabla \cdot \nabla = \nabla_x^2 + \nabla_y^2 + \nabla_z^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{49}$$

is the *Laplace operator* or *Laplacian*, which plays a major role in all field theory.

Poisson's equation is a single second order partial differential equation, and it follows from the derivation that the non-local potential integral over the mass distribution (31) must be a solution to Poisson's equation. The linearity of Poisson's equation guarantees that if Φ_1 is a particular solution then the most general solution is of the form $\Phi = \Phi_0 + \Phi_1$ where Φ_0 is an arbitrary solution to *Laplace's equation*, $\nabla^2\Phi_0 = 0$. The actual solution selected in a particular problem depends on the boundary conditions. To eliminate the arbitrary solution to Laplace's equation in the integral (31), we have imposed the boundary condition that the potential vanishes at infinity. Poisson's equation can, however, also be used to find the field in cases where this boundary condition cannot be imposed.

Example 4 [Universe with constant mass density]: If the universe were uniformly filled with matter of constant density, $\rho(\boldsymbol{x}) = \rho_0$, we would have to solve $\nabla^2\Phi = 4\pi G\rho_0$. It is easy to verify explicitly that a particular solution to this equation is

$$\Phi = \frac{2}{3}\pi G\rho_0 \, |\boldsymbol{x}|^2 = \frac{2}{3}\pi G\rho_0 \left(x^2 + y^2 + z^2\right), \tag{50}$$

yielding the gravitational acceleration field

$$\boldsymbol{g} = -\frac{4}{3}\pi G\rho_0 \boldsymbol{x}. \tag{51}$$

This gravitational field always points towards the origin of the coordinate system, which is thus imbued with an unphysical preferred status, not present in the specification of the problem. Although this example may seem farfetched, it was shown in section CM-12.6 that precisely this field appears naturally in Newtonian cosmology, and that it in fact does not confer a special physical status to the origin of the coordinate system.

7 Gravitational energy

What is the gravitational energy of a planet or a star? Since the gravitational potential of a finite body is always negative and grows more negative the closer one gets to the body, one does not have to perform any work to make such a body grow. It is sufficient to throw material into the general vicinity of the body, and let gravity do the rest. Consequently, the total gravitational energy of a body is expected to be negative. Gravity is in this respect different from most of the other forces we meet in daily life, for example friction, where we have to perform work to get anything done. It does not cost us anything to make matter collapse gravitationally, quite the contrary, we get paid for it (usually the payment is heat). Matter is inherently unstable because of gravity, and this instability lies at the root of galaxy and star formation CM-[Chandrasekhar 1981] , and thus of everything that is.

Assembly work

In section 5 it was shown that the work required to move a small particle of mass m from spatial infinity, where the gravitational potential vanishes, to a point $\boldsymbol{x}$ where the potential takes the value $\Phi(\boldsymbol{x})$ is $m\Phi(\boldsymbol{x})$. After you have done this (negative) work, it is conserved in the potential energy of the particle. Imagine now that we wish to increase the mass density by an amount $\delta\rho(\boldsymbol{x})$. The total work required to assemble this extra mass by bringing each material particle in from spatial infinity is,

$$\delta W = \int \Phi \, \delta\rho \, dV \tag{52}$$

where the integral runs over all space, or at least the volume inside which the mass density is non-vanishing. The added mass density $\delta\rho$ will also change the potential by a small amount $\delta\Phi$, but its contribution to the work will be of higher order in $\delta\rho$, and can be disregarded. If no other energy is added to or removed from the body, the above work will be stored in the body as gravitational energy.

In an *external* potential Φ, not originating from or influenced by the mass distribution itself, the total work of assembly becomes,

$$W = \int \Phi_{\rm ext} \, \rho \, dV. \tag{53}$$

In a constant gravitational field $\boldsymbol{g}_0$ where $\Phi_{\rm ext} = -\boldsymbol{x} \cdot \boldsymbol{g}_0$ we get,

$$W = -\boldsymbol{x}_{\rm M} \cdot M \boldsymbol{g}_0, \tag{54}$$

where $\boldsymbol{x}_{\rm M}$ is the center of mass (3).

The potential energy of a body in a constant external gravitational field is equivalent to the potential energy of a point particle with the total mass situated at the center of mass.

Gravitational self-energy

When a mass distribution is assembled in its own field, it is intuitively rather clear that each particle moved in from infinity on average meets only half the field of the final body. Hence the energy is expected to be only half of what it would be in an external field.

To show that there is indeed such a factor $\frac{1}{2}$ we shall employ a frequently used trick. Let us imagine that we build up the mass distribution ρ in such a way that at any given time, the distribution will be $\lambda\rho$ where $0 < \lambda < 1$. Since the potential is linear in the mass density, the current potential will also be the same fraction $\lambda\Phi$ of the final potential Φ. Increasing the fraction of the mass distribution by $\delta\lambda$ will then according to (52) cost an amount of work,

$$\delta W = \int (\lambda\Phi)(\delta\lambda \, \rho) \, dV = \lambda\delta\lambda \int \Phi\rho \, dV. \tag{55}$$

Integrating over λ from 0 to 1, we get the total amount of work we have to perform in building up the complete mass distribution from scratch,

$$\boxed{W = \frac{1}{2}\int \Phi\rho \, dV = -\frac{1}{2} G \iint \frac{\rho(\boldsymbol{x})\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} \, dV \, dV'.} \tag{56}$$

where we in the last step have introduced the potential (31) to expose the symmetry of the interaction of the mass density with itself, and that the work is always negative. The total gravitational *self-energy* stored in the self-interacting mass distribution equals the assembly work $\mathcal{E} = W$.

Example 5 [Spherical planet with constant density]: A planet of radius a with constant density ρ_0 and mass $M_0 = \frac{4}{3}\pi a^3 \rho_0$ has the potential (35). Carrying out the integral we find the total gravitational self-energy of the planet,

$$\mathcal{E} = -\frac{3}{5}\frac{GM_0^2}{a}. \tag{57}$$

In spite of the primitive nature of the model, we shall use this expression for an order of magnitude estimate of the gravitational energy of a planet or star.

For the Moon we get $\mathcal{E} = -1.2 \times 10^{29}$ J, for Earth $\mathcal{E} = -2.3 \times 10^{32}$ J and for the Sun $\mathcal{E} = -2.3 \times 10^{41}$ J. Already in 1854 Hermann Helmholtz proposed that the Sun's energy production stemmed from gravitational energy converted into heat during its assembly and subsequent contraction. Since the Sun's steady energy output is 3.85×10^{26} W, it would only last for 5.9×10^{14} s or merely about 19 million years before the stored energy would have been used up. The paradox was resolved in the 1930s with the modern understanding of the thermonuclear processes responsible for the Sun's energy production.

* Field energy density

Until now we have calculated the total gravitational self-energy from the non-local interaction of the mass density with itself. It obviously only receives contributions from regions where the mass density is non-zero. Interestingly, it is possible to transform it into an expression involving only the field strength $\boldsymbol{g}$ which is non-vanishing over all of space.

We demonstrate this by making use of the nabla-relationship,

$$\nabla \cdot (\Phi \boldsymbol{g}) = \Phi \nabla \cdot \boldsymbol{g} + (\boldsymbol{g} \cdot \nabla)\Phi, \tag{58}$$

which is most easily proven by writing it explicitly out in coordinates. Integrating over a volume V and using Gauss' theorem CM-(C.15) on the left-hand side we obtain

$$\oint_S \Phi \boldsymbol{g} \cdot d\boldsymbol{S} = \int_V \nabla \cdot (\Phi \boldsymbol{g})\, dV = -4\pi G \int_V \Phi\rho\, dV - \int_V \boldsymbol{g}^2 dV,$$

where on the right-hand side we have used both $\nabla\Phi = -\boldsymbol{g}$ and $\nabla \cdot \boldsymbol{g} = -4\pi G\rho$. If we now let the volume V expand to include all of space, the left-hand side will tend towards zero for a spatially bounded mass distribution. For at large distance r we have $\Phi \sim 1/r$ and $\boldsymbol{g} \sim 1/r^2$, so that $\Phi\boldsymbol{g} \sim 1/r^3$, whereas the surface area expands only as r^2. The left-hand side thus goes to zero as $1/r$.

In the limit we may rewrite the gravitational energy (56) in the form

$$\mathcal{E} = -\frac{1}{8\pi G} \int \boldsymbol{g}^2\, dV, \tag{59}$$

which explicitly demonstrates that the gravitational self-energy of a body is always negative. It now seems that the gravitational energy is locally distributed over all of space with an *energy density*

$$\boxed{\epsilon = -\frac{\boldsymbol{g}^2}{8\pi G},} \tag{60}$$

which is non-vanishing even in regions of space completely devoid of matter. At the surface of the Earth, the gravitational energy density is a whopping −57 gigajoule per cubic meter.

As discussed in section 1.6, the question of whether there is *really* energy out there in empty space depends largely on your theoretical frame of mind. In classical Newtonian physics, rewriting the self-energy as an integral over an energy density is just another mathematical trick.

Example 6 [Spherical planet with constant density]: In the spherical case we use (24) and obtain

$$\mathcal{E} = -\frac{1}{2} G \int_0^\infty \frac{M(r)^2}{r^2}\, dr. \tag{61}$$

This integral always converges for a body of finite mass, i.e., provided $M(r) \to M_0$ for $r \to \infty$, even if it has no boundary. Inserting $M(r) = \frac{4}{3}\pi r^3 \rho_0$ for $r < a$ and M_0 for $r > a$, one immediately recovers (57).

Problems

1 Show that a satellite moving in a circular orbit around a spherical planet has velocity $v = \sqrt{-\Phi}$, where Φ is the gravitational potential in the satellite's orbit. Calculate the velocity of a satellite moving at ground level.

2 Arthur C. Clarke proposed (*Wireless World*, October 1945, pp 305–308) that communication satellites should be put into a circular equatorial orbit with revolution time equal to Earth's rotation period, so that the satellites would stay fixed over a point at the equator. Calculate the height of the orbit above the ground, also taking into account that the Earth circles the Sun in one year.

3 A comet consisting mainly of ice falls to Earth. **(a)** Estimate the minimum energy released in the fall per unit of mass. **(b)** Compare with the estimate of the energy needed to evaporate the comet.

4 A stone is set in free fall from rest through a mine shaft going right through the center of a non-rotating planet with constant density. **(a)** Calculate the speed with which the stone passes the center. **(b)** Calculate the time it takes to fall to the center.

5 A planet consists of two layers with constant mass density,

$$\rho(r) = \begin{cases} \rho_1 & r \le a_1 \\ \rho_2 & a_1 < r \le a \\ 0 & r > a \end{cases} . \tag{62}$$

(a) Calculate the strength of gravity and the potential. **(b)** Show that the strength of gravity at the boundary between the layers is greater than at the surface when

$$\frac{\rho_1 - \rho_2}{\rho_2} > \frac{a^2}{a_1(a + a_1)}. \tag{63}$$

Verify that this is fulfilled for the Earth.

* **6** Show by direct integration in a small spherical volume around the singularity in (8) that it gives a finite contribution to the integral.

* **7** Show that the mass density is a scalar field.

* **8** Show that the gravitational field is a vector field.

* **9** Show that gravitational field of a spherical body (24) may be derived by integration of (8).

* **10** A spherical planet has mass distribution of the form $\rho(r) = Ar^\alpha$ for $r \le a$. **(a)** Calculate the gravitational field strength and the potential inside the planet for this distribution. **(b)** For what values of α is the problem solvable with finite planet mass? **(c)** For what value of α does gravity grow stronger towards the center?

* **11** An "exponential star" has a mass density $\rho = \rho_0 e^{-r/a}$, where ρ_0 is the central mass density and a is the 'radius'. Calculate the gravitational field and potential.

* **12** **(a)** Calculate the gravitational potential and field from a mass distribution shaped like a very thin line (a model of a cosmic string) of length $2a$ with uniform mass λ per unit of length. **(b)** Calculate the behaviour of the potential at infinity orthogonally to the line. **(c)** Discuss what happens in the limit of $a \to \infty$.

∗ **13** **(a)** Write an expression for the gravitational potential from a mass distribution shaped like a very thin circular plate of radius a with uniform mass σ per unit of area (a model of the luminous matter of a galaxy). **(b)** Calculate the value of the potential along the central normal of the plate. **(c)** Calculate its form far from the disk. **(d)** What happens for $a \to \infty$?

14 Show that the corrections to the asymptotic potential (32) are of order $(a/r)^2$.

15 **(a)** Show that the potential of a spherical mass distribution satisfies

$$g(r) = -\frac{d\Phi(r)}{dr} \qquad\qquad \Phi(r) = \int_r^\infty g(r')dr' \tag{64}$$

(b) Show that

$$\Phi(r) = -G\frac{M(r)}{r} - 4\pi G\int_r^\infty s\rho(s)\,ds. \tag{65}$$

What is the significance of the last term?

Answers

1 The centripetal acceleration in a circular orbit must equal the force of gravity, $v^2/r = GM/r^2$ leading to $v = \sqrt{GM/r} = \sqrt{-\Phi} = \sqrt{g_0 a^2/r}$. At ground level the velocity becomes $v = v_{\mathrm{esc}}/\sqrt{2} = 7.9\ \mathrm{km\ s^{-1}}$ where $v_{\mathrm{esc}} = 11.2\ \mathrm{km\ s^{-1}}$ is the escape velocity.

2 Earth's true rotation period $T = T_0 * 365/366$ is a bit shorter than $T_0 = 24$ hours because the orbital motion adds one full revolution in one year. Taking $v = \Omega r$ where $\Omega = 2\pi/T$ we find from the equality of centripetal acceleration and gravity that

$$r\Omega^2 = g_0\frac{a^2}{r^2}. \tag{66}$$

which solved for r/a yields

$$\frac{r}{a} = \left(\frac{g_0}{a\Omega^2}\right)^{1/3} \approx 6.613. \tag{67}$$

The orbit height is $h = r - a \approx 5.613a \approx 35,800$ km.

3

(a) Minimal kinetic energy: $\frac{1}{2}v_{\mathrm{esc}}^2 \approx 63\ (\mathrm{km\ s})^{-2} = 63 \times 10^6\ \mathrm{J\ kg^{-1}}$.

(b) Melting, heating and evaporating ice about $\approx 3.6 \times 10^6\ \mathrm{J\ kg^{-1}}$.

4 Energy conservation: $\frac{1}{2}\dot{r}^2 + \Phi(r) = \Phi(a)$. Use (35).

(a) $v_0 = -\,\dot{r}|_{r=0} = a\sqrt{2(\Phi(a) - \Phi(0))} = a\sqrt{\frac{4}{3}\pi\rho_0 G} = \sqrt{g_0 a} = 7.9\ \mathrm{km\ s^{-1}}$.

(b) $t_0 = \int_0^a \dfrac{dr}{\sqrt{2(\Phi(a) - \Phi(r))}} = \int_0^a \dfrac{dr}{\sqrt{\frac{4}{3}\pi\rho_0 G(a^2 - r^2)}} = \dfrac{\pi a}{2v_0} = 1267$ s.

5 **(a)** From (24) we get

$$g(r) = -\frac{4}{3}\pi G \begin{cases} r\rho_1 & r \le a_1 \\ \frac{a_1^3}{r^2}\rho_1 + \left(r - \frac{a_1^3}{r^2}\right)\rho_2 & a_1 < r \le a \\ \frac{a_1^3\rho_1 + (a^3 - a_1^3)\rho_2}{r^2} & r > a. \end{cases} \tag{68}$$

and from (34)

$$\Phi(r) = -\frac{2}{3}\pi G \begin{cases} (3a_1^2 - r^2)\rho_1 + 3(a^2 - a_1^2)\rho_2 & r \le a_1 \\ 2\frac{a_1^3}{r}\rho_1 + \left(3a^2 - r^2 - 2\frac{a_1^3}{r}\right)\rho_2 & a_1 \le r \le a \\ 2\frac{a_1^3}{r}\rho_1 + 2\frac{a^3 - a_1^3}{r}\rho_2 & r \ge a. \end{cases} \tag{69}$$

(b) It follows from $|g(a_1)| > |g(a)|$, that $a_1\rho_1 > (a_1^3\rho_1 + (a^3 - a_1^3)\rho_2)/a^2$ which may be rewritten in the form of the inequality (63). For the Earth the left-hand side becomes 1.42 and the right-hand side 1.18, so the inequality is fulfilled.

6 Cut out a small sphere $|\boldsymbol{x}' - \boldsymbol{x}| \le a$ around the point $\boldsymbol{x}$. Let a be so small that $\rho(\boldsymbol{x}')$ does not vary appreciably within this sphere. Then we get the contribution to gravity from the small sphere

$$\Delta \boldsymbol{g}(\boldsymbol{x}) = -G \int_{|\boldsymbol{x}'-\boldsymbol{x}|\le a} \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3}\, \rho(\boldsymbol{x}')\, dv' \approx -G\rho(\boldsymbol{x}) \int_{|\boldsymbol{x}'-\boldsymbol{x}|\le a} \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3}\, dv' = \boldsymbol{0}.$$

The last integral vanishes because of the spherical symmetry (it is a vector with no direction to point in).

7 A small volume is invariant under a rotation $dV' = dV$ and so is the amount of mass contained in it, $dM' = dM$. By the definition (1) we have $dM' = \rho'(\boldsymbol{x}')dV'$ and $dM = \rho(\boldsymbol{x})dV$ and from that $\rho'(\boldsymbol{x}') = \rho(\boldsymbol{x})$.

8 By the definition (9) we have $d\boldsymbol{\mathcal{F}}' = \boldsymbol{g}'(\boldsymbol{x}')\, dM'$ and $d\boldsymbol{\mathcal{F}} = \boldsymbol{g}(\boldsymbol{x})dM$. The force on a small volume is a vector and transforms according to $d\boldsymbol{\mathcal{F}}' = \mathbf{A} \cdot d\boldsymbol{\mathcal{F}}$ where $\mathbf{A}$ is the rotation matrix, and the mass element is invariant $dM' = dM$. From this we get $\boldsymbol{g}'(\boldsymbol{x}') = \mathbf{A} \cdot \boldsymbol{g}(\boldsymbol{x})$.

9 Multiplying (8) by $\boldsymbol{e}_r = \boldsymbol{x}/r$ and using (22) one gets

$$g(r) = -G \int_{|\boldsymbol{x}'|\le a} \frac{\boldsymbol{x} \cdot (\boldsymbol{x} - \boldsymbol{x}')}{r\, |\boldsymbol{x} - \boldsymbol{x}'|^3} \rho(\boldsymbol{x}')\, dv'.$$

Introducing $s = |\boldsymbol{x}'|$ and the angle θ between $\boldsymbol{x}$ and $\boldsymbol{x}'$, so that $dv' = 2\pi \sin\theta d\theta s^2 ds$, this becomes

$$g(r) = -2\pi G \int_0^a \rho(s) s^2 ds \int_{-1}^{1} d\cos\theta \frac{r - s\cos\theta}{(r^2 + s^2 - 2rs\cos\theta)^{\frac{3}{2}}}.$$

Integrating over $u = \cos\theta$ one obtains

$$\int_{-1}^{1} du \frac{r - su}{(r^2 + s^2 - 2rsu)^{\frac{3}{2}}} = -\frac{\partial}{\partial r} \int_{-1}^{1} du \frac{1}{\sqrt{r^2 + s^2 - 2rsu}}$$

$$= \frac{\partial}{\partial r} \left[\frac{\sqrt{r^2 + s^2 - 2rsu}}{rs} \right]_{u=-1}^{1} = \frac{\partial}{\partial r} \frac{|r - s| - (r + s)}{rs}$$

$$= -2\frac{\partial}{\partial r} \begin{cases} \frac{1}{r} & r > s \\ \frac{1}{s} & r < s \end{cases} = \begin{cases} \frac{2}{r^2} & r > s \\ 0 & r < s \end{cases}$$

which leads to the desired result (24).

10

(a) $g(r) = -4\pi G \frac{A}{3+\alpha} r^{1+\alpha}$, $\Phi(r) = 4\pi G \frac{A}{2+\alpha}\left(\frac{r^{2+\alpha}}{3+\alpha} - a^{2+\alpha}\right)$.

(b) $\alpha > -3$.

(c) $-3 < \alpha < -1$.

11 Use equation (24). Setting $u = r/a$ one gets

$$M(r) = \int_0^r \rho(s) 4\pi s^2\, ds = 4\pi\rho_0 \int_0^r e^{-s/a} s^2\, ds = 4\pi\rho_0 a^3 (2 - (2 + 2u + u^2)e^{-u}).$$

Similarly, using (65) one finds

$$\int_r^\infty s\rho(s)\, ds = \rho_0 \int_r^\infty s e^{-s/a}\, ds = \rho_0 a^2 (1+u) e^{-u}$$

and from this

$$\Phi = -\frac{4\pi G \rho_0 a^3}{r}(2(1 - e^{-u}) - u e^{-u}).$$

12 Line distribution $\rho dv = \lambda dz$ along the z-axis. Put $r = \sqrt{x^2 + y^2}$.

(a) Substitute $z' = z - r \sinh\psi$

$$\Phi = -G \int_{-a}^{a} \frac{\lambda dz'}{\sqrt{r^2 + (z - z')^2}} = -G\lambda(\sinh^{-1}\frac{z+a}{r} - \sinh^{-1}\frac{z-a}{r})$$

where $\sinh^{-1} u = \log(u + \sqrt{u^2+1})$ is the inverse hyperbolic sine. Then one gets

$$g_z = -\frac{\partial\Phi}{\partial z} = G\lambda\left(\frac{1}{\sqrt{(r^2 + (z+a)^2}} - \frac{1}{\sqrt{(r^2 + (z-a)^2}}\right)$$

$$g_r = -\frac{\partial\Phi}{\partial r} = -G\lambda\frac{1}{r}\left(\frac{z+a}{\sqrt{(r^2 + (z+a)^2}} - \frac{z-a}{\sqrt{(r^2 + (z-a)^2}}\right).$$

(b) For $r \to \infty$: $\Phi \to -G\frac{2a\lambda}{r}$, $g_z \to -G 2a\lambda \frac{z}{r^2}$, $g_r \to -G\frac{2a\lambda}{r^2}$.

(c) For $a \to \infty$: $\Phi \to -2G\lambda \log\frac{a}{r}$, $g_z \to -2G\lambda\frac{z}{a^2}$, $g_r \to -\frac{2G\lambda}{r}$.

13 Use cylindrical coordinates (r, ϕ, z).

(a)

$$\Phi = -G\sigma \int_0^a s ds \int_0^{2\pi} d\phi \frac{1}{\sqrt{z^2 + r^2 + s^2 - 2rs\cos\phi}}.$$

(b) $\Phi = -2\pi G\sigma(\sqrt{z^2 + a^2} - |z|)$.

(c) $\Phi \to -G\frac{\sigma\pi a^2}{|z|}$.

(d) $\Phi \to -2\pi G(a - |z|)$ for $a \to \infty$.

14 Expand to first order in the small quantity $\boldsymbol{r}' = \boldsymbol{x}' - \boldsymbol{x}_M$,

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \approx \frac{1}{\sqrt{r^2 - 2\boldsymbol{r} \cdot \boldsymbol{r}'}} \approx \frac{1}{r}\left(1 + \frac{\boldsymbol{r} \cdot \boldsymbol{r}'}{r^2} + \mathcal{O}\left(\frac{a^2}{r^2}\right)\right) \tag{70}$$

But $\int_V \boldsymbol{r}'\rho(\boldsymbol{x}')dV' = 0$ and the result follows.

15 **(a)** Calculate the negative gradient of $\Phi(r)$ using (29),

$$\boldsymbol{g} = -\nabla\Phi = -\frac{d\Phi}{dr}\nabla r = -\frac{d\Phi}{dr}\boldsymbol{e}_r. \tag{71}$$

we get $g(r) = -d\Phi/dr$. Integrate this relation, using that the potential must vanish at infinity.

(b) Perform a partial integration

$$\Phi(r) = -G\int_r^\infty \frac{M(s)}{s^2}\,ds = G\int_r^\infty M(s)\,d\left(\frac{1}{s}\right) = -G\frac{M(r)}{r} - 4\pi G\int_r^\infty s\rho(s)\,ds. \tag{72}$$

The last term secures the smoothness of the potential at the surface of the planet.

References

[1] J. C. Long, H. W. Chan, A. B. Churnside, E. A. Gulbis, M. C. M. Varney, and J. C. Price: *Upper limits to submillimeter-range forces from extra space-time dimensions*, Nature **421** (2003) 922–925.