Extra problems

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The following problem collection is organized by the chapter number, and the problem numbers continue the list found at the end of each chapter.

1 Continuous matter

Problem 1.5 The probability density of finding a gas molecule with velocity vector $\boldsymbol{v} = (v_x, v_y, v_z)$ is given by the Maxwell-Boltzmann distribution

$$p(v_x, v_y, v_z) = \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta(v_x^2+v_y^2+v_z^2)}, \qquad \text{with } \beta = \frac{m}{2kT}$$

where m is the mass of the molecule, k Boltzmann's constant and T the absolute temperature.

(a) Show that the distribution is normalized,

$$\int_{-\infty}^{\infty} p(v_x, v_y, v_z) dv_x dv_y dv_z = 1$$

(b) Show that the normalized probability distribution for the *molecular speed* is

$$p(v) = \left(\frac{\beta}{\pi}\right)^{3/2} 4\pi v^2 e^{-\beta v^2}, \qquad v \equiv |\boldsymbol{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

(c) Show that the average speed and average speed squared are

$$\langle v \rangle = \frac{2}{\sqrt{\pi\beta}}, \qquad \langle v^2 \rangle = \frac{3}{2\beta}.$$

(d) Show that the variance of the speed is

$$\sigma^2 \equiv \left\langle (v - \langle v \rangle)^2 \right\rangle = \frac{3\pi - 8}{2\pi\beta}$$

(e) Calculate $\langle v \rangle$, $\sqrt{\langle v^2 \rangle}$, and σ for air at normal temperature and pressure.

25 Gravity waves

Problem 25.8 The shallow-water fields given in Equation PCM-(25.33) are an approximation valid for $d \ll \lambda$. Although these velocity fields can be derived from the gradient of the velocity potential ($\boldsymbol{v} = \nabla\Psi$), they do not obey the equation of mass conservation for incompressible fluid ($\nabla \cdot \boldsymbol{v} = 0$). **(a)** Calculate the leading non-trivial correction to the velocity field in the shallow-wave approximation. **(b)** To what extent does this velocity field satisfy the divergence condition.

D Curvilinear coordinates

Problem D.2 Verify the stress tensor in cylindrical coordinates for an incompressible fluid

$$
\begin{aligned}
\sigma_{rr} &= -p + 2\eta \nabla_r v_r, \\
\sigma_{r\phi} = \sigma_{\phi r} &= \eta \left(\nabla_r v_\phi + \frac{\nabla_\phi v_r}{r} - \frac{v_\phi}{r} \right), \\
\sigma_{rz} = \sigma_{zr} &= \eta \left(\nabla_r v_z + \nabla_z v_r \right), \\
\sigma_{\phi\phi} &= -p + 2\eta \left(\frac{v_r}{r} + \frac{\nabla_\phi v_\phi}{r} \right), \\
\sigma_{\phi z} = \sigma_{z\phi} &= \eta \left(\nabla_z v_\phi + \frac{\nabla_\phi v_z}{r} \right), \\
\sigma_{zz} &= -p + 2\eta \nabla_z v_z.
\end{aligned}
$$

Problem D.3 Verify the stress tensor in spherical coordinates for an incompressible fluid

$$
\begin{aligned}
\sigma_{rr} &= -p + 2\eta \nabla_r v_r, \\
\sigma_{r\theta} = \sigma_{\theta r} &= \eta \left(\nabla_r v_\theta + \frac{\nabla_\theta v_r}{r} - \frac{v_\theta}{r} \right), \\
\sigma_{r\phi} = \sigma_{\phi r} &= \eta \left(\nabla_r v_\phi + \frac{\nabla_\phi v_r}{r \sin\theta} - \frac{v_\phi}{r} \right), \\
\sigma_{\theta\theta} &= -p + 2\eta \left(\frac{\nabla_\theta v_\theta}{r} + \frac{v_r}{r} \right), \\
\sigma_{\theta\phi} = \sigma_{\phi\theta} &= \eta \left(\frac{\nabla_\theta v_\phi}{r} + \frac{\nabla_\phi v_\theta}{r \sin\theta} - \frac{v_\phi}{r \tan\theta} \right), \\
\sigma_{\phi\phi} &= -p + 2\eta \left(\frac{\nabla_\phi v_\phi}{r \sin\theta} + \frac{v_\theta}{r \tan\theta} + \frac{v_r}{r} \right).
\end{aligned}
$$