Stokes waves

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In 1847 Stokes published a seminal paper [1] on nonlinear gravity waves that generalize the linear small-amplitude harmonic waves at constant depth discussed in Section 25.3 at page CM-424. For many years these *Stokes waves* stood as *the* model for nonlinear waves, even if they today are known to suffer from small frequency instabilities [Mei 1989].

1 Periodic, permanent line waves

Gravity waves in a liquid ("water") are essentially only interesting for large Reynolds number, where viscosity plays an insignificant role. Compression plays likewise only a minor role, when the material velocity in a wave is small compared to the sound velocity in the liquid, which it normally is. Consequently, the liquid may be assumed to be an incompressible Euler liquid (see Chapter CM-13). Furthermore, if the waves have been created from water originally at rest without vorticity, the flow may also be assumed to be irrotational at all times.

Incompressible, inviscid, and irrotational water

Under these conditions, the two-dimensional flow in a line wave may be derived from a velocity potential $\Psi(x, y, t)$ that satisfies Laplace's equation (see page CM-220),

$$v_x = \nabla_x \Psi, \qquad v_y = \nabla_y \Psi, \qquad \left(\nabla_x^2 + \nabla_y^2\right) \Psi = 0. \tag{1}$$

Any solution to the Laplace equation yields a possible two-dimensional flow.

With constant gravity g_0 pointing along the negative y-axis, the undisturbed surface of the sea corresponds to $y = 0$. Let the surface of the wave be described by a function $y = h(x, t)$. A liquid particle sitting right at the surface (or rather just below) must follow the motion of the surface. In a small time interval δt, the particle is displaced horizontally by $\delta x = v_x \delta t$ and vertically $\delta y = v_y \delta t$, so that we must have $h(x + \delta x, t + \delta t) = h(x, t) + \delta y$. Expanding to first order in the small quantities, we get the kinematic surface condition,

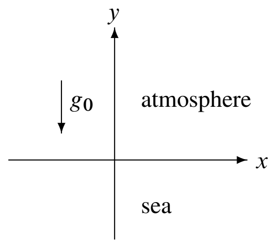
$$\frac{\partial h}{\partial t} + v_x \nabla_x h = v_y \qquad \text{for } y = h. \tag{2}$$

The pressure in the liquid is given by Equation (13.34) at page CM-220. At the liquid surface against vacuum (or air) the pressure must be constant (in the absence of surface tension), and we get the dynamic surface condition

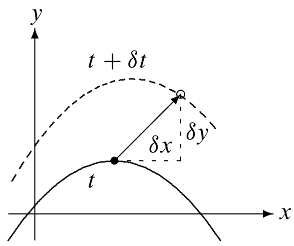
$$g_0 y + \frac{1}{2}\left(v_x^2 + v_y^2\right) + \frac{\partial \Psi}{\partial t} = \text{const} \qquad \text{for } y = h \tag{3}$$

The constant is rather unimportant, since it can be eliminated by shifting the origin of y.

Finally, we must require that the $v_y = 0$ for $y = -d$, where d is the constant depth of the undisturbed ocean.



In the two-dimensional flat-Earth coordinate system, constant gravity points along the negative y-axis, and the surface of the undisturbed sea is at $y = 0$. The z-axis points out of the paper.



A fluid particle at the surface must follow the motion of the surface from time t to time $t + \delta t$.

Permanence

A line wave is said to be *permanent* or *stationary* if it progresses along the x-axis with unchanging shape and constant celerity c, such that all surface features move with the same velocity c. The surface shape can therefore only depend on the combination $x - ct$, and we expect that this is also the case for the velocity potential, that is, $\Psi = \Psi(x - ct, y)$. We may consequently replace all time derivatives by x-derivatives according to the rule $\partial/\partial t \to -c\partial/\partial x$, and afterwards put $t = 0$, so that the velocity potential $\Psi(x, y)$ and the derived fields only depend on x and y. The surface boundary conditions may thus be written

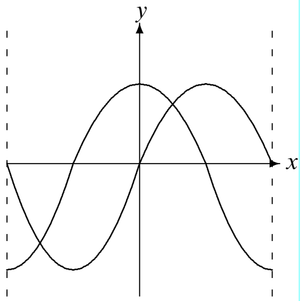
$$v_y = (v_x - c)h' \qquad \text{for} \quad y = h, \tag{4}$$
eSWkinematic

where $h' = dh/dx$, and

$$g_0 h = c v_x - \frac{1}{2}\left(v_x^2 + v_y^2\right) + \text{const}, \qquad \text{for} \quad y = h. \tag{5}$$
eSWdynamic

The bottom condition is as before $v_y = 0$ for $y = -d$.

Periodicity and symmetry



Even and odd periodic functions.

Stokes also assumed that the waves are periodic with spatial period λ,

$$h(x + \lambda) = h(x), \qquad \Psi(x + \lambda, y) = \Psi(x, y), \tag{6}$$

so that the nonlinear waves in this respect are generalizations of the linear ones.

The Laplace equation is trivially invariant under a change of sign in x. Because of its linearity, its solutions may accordingly be classified as symmetric (even) or antisymmetric (odd) under a change of sign of x. There is, however, no guarantee that a solution satisfying the nonlinear boundary conditions might not be an asymmetric linear combination of odd and even velocity potentials. We shall later return to this question, but for now we assume that $\Psi(x, y)$ may be chosen odd in x,

$$\Psi(-x, y) = -\Psi(x, y). \tag{7}$$

The boundary conditions then imply that v_x is even and v_y odd, and consequently that the surface height h must be even. Notice that the antisymmetry of $\Psi(x, y)$ also implies that all even derivatives at $x = 0$ must vanish. Conversely, all odd derivatives of $h(x)$ must vanish, implying that $x = 0$ is an extremum of $h(x)$.

Choice of units

The parameters describing periodic line waves are the celerity c, wavelength λ (or wavenumber $k = 2\pi/\lambda$), depth d, amplitude a, and strength of gravity g_0. The only dimensionless variables are ka, kd, and

$$G = \frac{g_0}{kc^2}. \tag{8}$$
eSWdispersionG

For general reasons G must be a function of ka and kd (dispersion relation). In the linear approximation CM-(25.28) we have $G = \coth kd$.

To simplify the formalism, we shall choose units of space and time so that,

$$k = 1, \qquad c = 1. \tag{9}$$

This implies $\lambda = 2\pi$, and that Equation (8) simplifies to $g_0 = G$.

Fourier expansion

The periodicity and symmetry permit us to expand the surface height into even Fourier components,

$$h = \sum_{n=1}^{\infty} H_n \cos nx, \tag{10}$$

where the H_n may depend on a and d. Although an even function would permit a term H_0, such a term is absent here because of mass conservation in the wave shape, which requires that $\int_{-\pi}^{\pi} h(x)\,dx = 0$. The series converges provided it is square integrable, that is, $\int_{-\pi}^{\pi} h(x)^2\,dx = \pi \sum_{n=1}^{\infty} H_n^2 < \infty$.

The periodicity and oddness of the velocity potential Ψ guarantees likewise that it may be expanded into odd Fourier components of the form $\Psi \sim f_n(y) \sin nx$ that must satisfy the Laplace equation, which here takes the form $f_n'' = n^2 f_n$. The general solution is $f_n = A_n \cosh ny + B_n \sinh ny$, but since each Fourier component of the vertical velocity $v_y \sim f_n'(y) \sin nx$ must vanish for $y = -d$, we must require $f_n'(-d) = 0$, and the solution becomes $f_n \sim \cosh n(y + d)$. In non-dimensional form we may thus write,

$$\Psi = \sum_{n=1}^{\infty} \Psi_n \frac{\cosh n(y+d)}{\sinh nd} \sin nx, \tag{11}$$

where the Ψ_n may depend on a and d. The denominator is introduced for later convenience (see the linear solution CM-(25.29)).

From the velocity potential we derive the velocity field components,

$$v_x = \sum_{n=1}^{\infty} n\Psi_n \frac{\cosh n(y+d)}{\sinh nd} \cos nx, \tag{12}$$

$$v_y = \sum_{n=1}^{\infty} n\Psi_n \frac{\sinh n(y+d)}{\sinh nd} \sin nx. \tag{13}$$

Using trigonometric relations, the y-dependent factors may also be written

$$f_n(y) \equiv \frac{\cosh n(y+d)}{\sinh nd} = \sinh ny + C_n \cosh ny, \tag{14}$$

$$g_n(y) \equiv \frac{\sinh n(y+d)}{\sinh nd} = \cosh ny + C_n \sinh ny, \tag{15}$$

where $C_n = \coth nd$ contain all the dependence on depth d. Notice that $f_n' = n g_n$ and $g_n' = n f_n$.

Truncation

Having solved the field equations and implemented the bottom boundary condition, we must now apply the surface boundary conditions to determine the unknown coefficients Ψ_n and H_n. A natural approximation consists in truncating the infinite series at $n = N$, that is, dropping all terms of harmonic order $n > N$. Here we shall only investigate the linear first order approximation and the second order nonlinear correction to it.

Linear approximation

For $N = 1$ we have

$$h = H_1 \cos x, \tag{16}$$

and

$$v_x = \Psi_1 f_1(y) \cos x, \qquad v_y = \Psi_1 g_1(y) \sin x. \tag{17}$$
eSWlinear

Inserted into the surface boundary condition (4) and (5), we get in the linear approximation $v_y = -h'$ and $Gh = v_x + \text{const}$ for $y = 0$ (because setting $y = h$ would generate second order harmonics), and we arrive at

$$\Psi_1 = H_1, \qquad GH_1 = \Psi_1 C_1, \tag{18}$$

while const $= 0$ in this approximation. Eliminating Ψ_1 we obtain

$$G = C_1, \qquad \text{or} \qquad c^2 = \frac{g_0}{kG} = \frac{g_0}{k} \tanh kd \tag{19}$$
eSWcelerity

which is simply the expression CM-(25.28) for the celerity of linear waves. The first-order coefficient H_1 is clearly a free parameter, and usually one defines $H_1 = \Psi_1 = a$ where a is the linear wave amplitude.

Second order approximation

For $N = 2$ we have

$$h = H_1 \cos x + H_2 \cos 2x, \tag{20}$$

and

$$v_x = \Psi_1 f_1(y) \cos x + 2\Psi_2 f_2(y) \cos 2x, \tag{21}$$
eSWvelocities

$$v_y = \Psi_1 g_1(y) \sin x + 2\Psi_2 g_2(y) \sin 2x. \tag{22}$$

Setting $y = h$ and expanding $f_n(h)$ and $g_n(h)$ in powers of h, we only need to keep the first-order harmonic term in the first terms and the zeroth-order term ($h = 0$) in the second. In other words, for $y = h$,

$$v_x = \Psi_1 (C_1 + H_1 \cos x) \cos x + 2\Psi_2 C_2 \cos 2x, \tag{23}$$

$$v_y = \Psi_1 (1 + C_1 H_1 \cos x) \sin x + 2\Psi_2 \sin 2x. \tag{24}$$

Inserted into the kinematic boundary condition (4), we get

$$\begin{aligned} &\Psi_1 (1 + C_1 H_1 \cos x) \sin x + 2\Psi_2 \sin 2x \\ &\quad = -(-1 + \Psi_1 C_1 \cos x) H_1 \sin x + 2H_2 \sin 2x, \end{aligned}$$

where all terms of harmonic order 3 and higher have been dropped. Using that $\sin x \cos x = \frac{1}{2} \sin 2x$, we arrive at the equations,

$$\Psi_1 = H_1, \qquad \Psi_2 = H_2 - \frac{1}{2} C_1 H_1^2, \tag{25}$$
eSWkinematic1

by matching the $\sin x$ and $\sin 2x$ terms on both sides.

Finally, the dynamic boundary condition (5) becomes to second harmonic order

$$G\left(H_1 \cos x + H_2 \cos 2x\right) = \Psi_1(C_1 + H_1 \cos x)\cos x + 2\Psi_2 C_2 \cos 2x$$
$$- \frac{1}{2}\left((\Psi_1 C_1 \cos x)^2 + (\Psi_1 \sin x)^2\right) + \text{const.}$$

Using that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, and matching the harmonic terms, we find,

$$GH_1 = \Psi_1 C_1, \qquad GH_2 = 2\Psi_2 C_2 + \frac{1}{2}\Psi_1 H_1 - \frac{1}{4}\Psi_1^2(C_1^2 - 1), \tag{26}$$

where the zeroth-order (constant) terms have been absorbed in the unknown constant.

Eliminating the Ψ's by means of (25), the solution becomes

$$G = C_1, \qquad H_2 = \frac{C_1^2 + 4C_1C_2 - 3}{4(2C_2 - C_1)} H_1^2, \qquad \Psi_2 = \frac{3(C_1^2 - 1)}{4(2C_2 - C_1)} H_1^2. \tag{27}$$

Using that $C_2 = \frac{1}{2}\left(C_1 + C_1^{-1}\right)$, this simplifies to:

$$G = C_1, \qquad H_2 = \frac{1}{4} C_1 \left(3C_1^2 - 1\right) H_1^2, \qquad \Psi_2 = \frac{3}{4} C_1 \left(C_1^2 - 1\right) H_1^2, \tag{28}$$

with the celerity again given by (19), and $C_1 = \coth d$. Notice that H_1 is again a free variable, so we shall again put $H_1 = \Psi_1 = a$ where a is the amplitude.

The surface shape becomes in the extremes,

$$h = a\cos x + \frac{1}{2}a^2 \cos 2x \qquad \text{for } d \to \infty \quad \text{(deep water)}, \tag{29}$$

$$h = a\cos x + \frac{3a^2}{4d^3}\cos 2x \qquad \text{for } d \to 0 \quad \text{(shallow water)}. \tag{30}$$

In deep water the corrections to the linear wave can be ignored for $a \ll 1$.

In shallow water the condition for ignoring the nonlinear term is that $a \ll d^3$ (in units of $\lambda = 2\pi$). The dimensionless ratio,

$$\boxed{\mathsf{Ur} = \frac{a}{k^2 d^3},} \tag{31}$$

is called the *Ursell number*[2], and the linear approximation is valid for $\mathsf{Ur} \ll 1$.

Example 1 [Tsunami]: A tsunami proceeds with wavelength $\lambda \approx 500$ km in an ocean of depth $d = 4$ km. The celerity becomes $c \approx 200\ \text{m s}^{-1}$, or about $700\ \text{km h}^{-1}$. For the linear wave theory to be valid we must have $a \ll k^2 d^3 = 10$ m, which is satisfied for $a = 1$ m, corresponding to $\mathsf{Ur} = 0.1$. One should note that there are, however, nonlinear secular changes to the wave shape taking place over very long distances that may become important.

Higher order approximations

In principle these methods may be used to obtain higher order harmonic corrections to the linear waves. As the analysis of second harmonics indicates, the calculation of the higher harmonics are liable to become increasingly complicated. This was also recognized by Stokes, and in a Supplement to his 1847 paper [1], he carried out a much more elegant series expansion of deep-water waves to fifth order. In the following section, we shall extend this calculation to 9'th order.

Stokes drift

In the strictly linear approximation, that is, to the first order in the amplitude a, the fluid particles move in elliptical orbits (see page CM-428). In higher order approximations, this is no more the case, and we shall see that the fluid particles beside periodic motion also perform a secular motion. We shall calculate this effect in the Lagrange representation, where the intuitive meaning is clearest.

A particle starting in (X, Y) at time $t = 0$ and found at (x, y) at time t is displaced horizontally by $u_x = x - X$ and vertically by $u_y = y - Y$. The particle velocity components are then determined from

$$\frac{du_x}{dt} = v_x(X - t + u_x, Y + u_y), \qquad \frac{du_y}{dt} = v_y(X - t + u_x, Y + u_y) \tag{32}$$

eSWdisplacementde

To lowest harmonic order we disregard the displacements on the right hand sides and integrate the velocities (21) to get

$$u_x = \Psi_1 f_1(Y) \sin(t - X), \qquad u_y = \Psi_1 g_1(Y) \cos(t - X). \tag{33}$$

Here the integration constants have been chosen such that the horizontal displacement vanishes for $t = X$ and the vertical displacement vanishes at the bottom $Y = -d$. Thus, to first harmonic order of approximation a fluid particle moves on an ellipse, which has horizontal major axis and vertical minor and flattens towards the bottom.

Next, we expand to first order in the displacements on the right-hand side of (32) to get

$$\frac{du_x}{dt} = v_x(X - t, Y) + u_x \frac{\partial v_x}{\partial X} + u_y \frac{\partial v_x}{\partial Y}, \tag{34}$$

$$\frac{du_y}{dt} = v_y(X - t, Y) + u_x \frac{\partial v_y}{\partial X} + u_y \frac{\partial v_y}{\partial Y}. \tag{35}$$

The leading terms are determined by (21) and the remainder from the lowest order displacements. To second harmonic order we find

$$\frac{du_x}{dt} = \Psi_1 f_1(y) \cos(t - X) + 2\Psi_2 f_2(Y) \cos 2(t - X) + \Psi_1^2 f_1(Y)^2 \sin^2(t - X) + \Psi_1^2 g_1(Y)^2 \cos^2(t - X). \tag{36}$$

$$\frac{du_y}{dt} = -\Psi_1 g_1(Y) \sin(t - X) + 2\Psi_2 g_2(Y) \sin 2(t - X). \tag{37}$$

Finally, integrating with respect to t and imposing the boundary conditions mentioned above, we get

$$u_x = \Psi_1 f_1(Y) \sin(t - X) + \left(\Psi_2 f_2(y) - \tfrac{1}{4}\Psi_1^2 \left(f_1^2(Y) - g_1^2(Y)\right)\right) \sin 2(t - X) + \tfrac{1}{2}\Psi_1^2 \left(f_1^2(y) + g_1^2(y)\right) (t - X), \tag{38}$$

$$u_y = \Psi_1 g_1(Y) \cos(t - X) + 2\Psi_2 g_2(y) \cos 2(t - X). \tag{39}$$

The term linear in $t - X$ shows that on top of its second order harmonic motion, the particle drifts along the x-axis with the *Stokes drift velocity*, which may be written:

$$\boxed{U = \tfrac{1}{2} c (ka)^2 \frac{\cosh 2k(Y + d)}{\sinh^2 kd}.} \tag{40}$$

where we have put back the dimensional constants. For shallow-water waves, this becomes $U \approx \frac{1}{2} c (a/d)^2$. In the tsunami example above, the drift velocity becomes a measly two centimeter per hour!

2 Stokes expansion

Instead of working in "laboratory" coordinates where the waves move with celerity $c = 1$ along the positive x-axis, it is most convenient to use comoving coordinates in which the wave surface is static and the flow below is steady.

Transforming the laboratory velocity field to a dimensionless comoving velocity field,

$$u_x = v_x - 1, \qquad u_y = v_y, \tag{41}$$

the surface boundary conditions (4) and (5) become

$$u_y = u_x h' \qquad \text{for } \; y = h, \tag{42}$$ eSWtop1

and

$$Gh = -\tfrac{1}{2}\left(u_x^2 + u_y^2\right) + C \qquad \text{for } \; y = h. \tag{43}$$ eSWtop2

where C is a constant and G is gravity when $c = k = 1$; see Equation (8).

The bottom conditions become

$$u_x \to -1, \qquad u_y \to 0 \qquad \text{for } \; y \to -\infty \tag{44}$$

The first condition simply expresses that the laboratory moves with velocity -1 relative to the comoving frame.

Conformal transformation

In comoving coordinates the dimensionless velocity potential is denoted $\phi(x, y)$, and the conjugate stream function $\psi(x, y)$. These functions satisfy the relations (see page CM-221)

$$u_x = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad u_y = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{45}$$ eNLconjugatepotentials

Combining the derivatives, we get the Laplace equations,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0, \tag{46}$$

expressing, respectively, incompressibility and irrotationality.

Rather than viewing x and y as independent variables, and $\phi = \phi(x, y)$ and $\psi = \psi(x, y)$ as dependent, Stokes proposed to view ϕ and ψ as independent variables, and $x = x(\phi, \psi)$ and $y = y(\phi, \psi)$ as dependent. To carry out the transformation from Cartesian coordinates (x, y) to the curvilinear coordinates (ϕ, ψ), we first solve the differential forms,

$$d\phi = u_x dx + u_y dy, \qquad d\psi = u_x dy - u_y dx, \tag{47}$$

and find,

$$dx = \frac{u_x d\phi - u_y d\psi}{u_x^2 + u_y^2}, \qquad dy = \frac{u_y d\phi + u_x d\psi}{u_x^2 + u_y^2}. \tag{48}$$

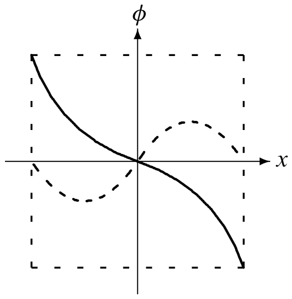
From these we read off

$$\frac{\partial x}{\partial \phi} = \frac{\partial y}{\partial \psi} = \frac{u_x}{u_x^2 + u_y^2}, \qquad \frac{\partial y}{\partial \phi} = -\frac{\partial x}{\partial \psi} = \frac{u_y}{u_x^2 + u_y^2}. \tag{49}$$ ePWderivs

Combining the derivatives, we arrive at

$$\frac{\partial^2 x}{\partial \phi^2} + \frac{\partial^2 x}{\partial \psi^2} = 0, \qquad \frac{\partial^2 y}{\partial \phi^2} + \frac{\partial^2 y}{\partial \psi^2} = 0. \tag{50}$$

Thus, the Cartesian coordinates x and y are also solutions to the Laplace equation in the curvilinear coordinates ϕ and ψ.



Periodicity in ϕ and x. Notice that both ϕ and x continue into the diagonally adjacent boxes while $x + \phi$ is truly periodic (dashed) in x.

Periodicity and symmetry

For periodic line waves with $k = 1$ and period $\lambda = 2\pi$, the dimensionless velocity field must obey

$$u_x(x + 2\pi, y) = u_x(x, y), \qquad u_y(x + 2\pi, y) = u_y(x, y). \tag{51}$$

But since $u_x \to -1$ for $y \to -\infty$, it follows from (45) that $\phi \to -x$ and $\psi \to -y$ in this limit, so that horizontal periodicity along x can only be imposed on $\phi + x$ and ψ:

$$\phi(x + 2\pi, y) = \phi(x, y) - 2\pi, \qquad \psi(x + 2\pi, y) = \psi(x, y). \tag{52}$$

Solving these equations for x and y, it follows that, $x(\phi, \psi) + \phi$ and $y(\phi, \psi)$ are also periodic in ϕ with period 2π.

We shall as before require that $\phi(x, y)$ is odd in x and $\psi(x, y)$ is even,

$$\phi(-x, y) = -\phi(x, y), \qquad \psi(-x, y) = \psi(x, y), \tag{53}$$

which translates into $x(\phi, \psi)$ being odd in ϕ and $y(\phi, \psi)$ even.

Using the periodicity and oddness we may expand $x + \phi$ into a sum of harmonic Fourier components of the form $F_n(\psi) \sin n\phi$. The Laplace equation requires $F_n''(\psi) = n^2\psi$, and using that $\psi \to +\infty$ for $y \to -\infty$ it follows that $F_n = -A_n e^{-n\psi}$ where the coefficients A_n are constants. Using the relations $\partial x/\partial\phi = \partial y/\partial\psi$ and $\partial x/\partial\psi = -\partial y/\partial\phi$ to determine y, we arrive at

$$x = -\phi - \sum_{n=1}^{\infty} A_n e^{-n\psi} \sin n\phi, \qquad y = -\psi + \sum_{n=1}^{\infty} A_n e^{-n\psi} \cos n\phi, \tag{54}$$

where we have defined $A_n = 0$ for $n < 1$ to avoid writing $n \geq 1$ in the sums.

Boundary conditions

At the surface of the water, $y = h(x)$, the stream function $\psi(x, h(x))$ is constant:

$$d\psi(x, h(x)) = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y} h'(x) = -u_y + u_x h'(x) = 0, \tag{55}$$

because of the boundary condition (42). Since a constant value of ψ can be absorbed into the A_n, we are free to choose $\psi = 0$ at the surface, and we obtain a parametric representation of the surface,

$$\boxed{x(\phi) = -\phi - \sum_{n=1}^{\infty} A_n \sin n\phi, \qquad y(\phi) = \sum_{n=1}^{\infty} A_n \cos n\phi.} \tag{56}$$

eSWimplicit

The wave height $h(x)$ must satisfy the relation

$$h(x(\phi)) = y(\phi). \tag{57}$$

for all ϕ.

From (49) we obtain,

$$S(\phi) \equiv \left(\frac{\partial x}{\partial\phi}\right)^2 + \left(\frac{\partial y}{\partial\phi}\right)^2 \Bigg|_{\psi=0} = \frac{1}{u_x^2 + u_y^2}\Bigg|_{y=h(x)}, \tag{58}$$

so that the dynamic condition (43) may be written:

$$F = (F + K - 2y(\phi))S(\phi), \tag{59}$$

where $F = 1/G$ and $K = (2C - 1)F$ are "constants" (that may depend on the amplitude). The constant F determines the dispersion relation,

$$c^2 = \frac{g_0}{k} F(ka), \tag{60}$$

expressed in proper dimensional parameters.

The implicit representation (56) of $y = h(x)$ through ϕ does not guarantee that the average value of h vanishes. A quick calculation yields the average

$$h_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x)\, dx = \frac{1}{2\pi} \int_{\pi}^{-\pi} y(\phi) \frac{dx(\phi)}{d\phi}\, d\phi = \frac{1}{2} \sum_{n=1}^{\infty} n A_n^2. \tag{61}$$

This will be used to plot the shape of the wave.

Truncation

What remains is to determine the coefficients A_n. As in the previous section, the boundary conditions permit us to determine all the A_n for $n \geq 2$ in terms of the first $A_1 = b$. We shall later find the relation between b and the amplitude a of the first-order harmonic in x. Since

$$2 \cos n\phi \cos m\phi = \cos(n + m)\phi + \cos(n - m)\phi, \tag{62}$$

$$2 \sin n\phi \sin m\phi = \cos(n - m)\phi - \cos(n + m)\phi, \tag{63}$$

it follows that the product $A_n A_m$ will be associated with harmonic functions up to order $n+m$.

Consequently, A_n can only depend on b^n and higher powers, so that we may write

$$A_n = \sum_{m=n}^{\infty} A_{nm} b^m. \tag{64}$$

Similarly, we shall put

$$F = 1 + \sum_{n=1}^{\infty} F_n b^n, \qquad K = \sum_{n=1}^{\infty} K_n b^n, \tag{65}$$

where zeroth order values $F = 1$ and $K = 0$ have been chosen so that Equation (59) is fulfilled in zeroth order (with $S = 1$). It is now "merely" a question of matching powers and trigonometric functions on both sides of this equation to determine the unknown coefficients.

Truncating the series by dropping all powers higher than b^N, we obtain from (59) an iterative scheme that successively determines the highest order coefficients (F_N, K_N, and $A_{n,N}$ with $n = 1, \cdots, N$) from the lower order coefficients already known. To simplify the analysis we shall assume that the coefficients F_n and K_n are only nonzero for even n whereas A_{nm} are only nonzero for n and m being either both even or both odd. These assumptions may be written

$$F_{\text{odd}} = K_{\text{odd}} = A_{\text{even,odd}} = A_{\text{odd,even}} = 0, \tag{66}$$

and will be justified by the solution.

First order ($N = 1$)

Defining $A_1 = b$, that is, $A_{11} = 1$ and $A_{1,n} = 0$ for $n > 1$, we have

$$x = -\phi - b\sin\phi + \mathcal{O}(b), \qquad y = b\cos\phi + \mathcal{O}(b) \tag{67}$$

Equation (59) us automatically fulfilled (since by assumption $F_1 = K_1 = 0$).

From Equation (61), we find the height average

$$h_0 = \tfrac{1}{2}b^2 + \mathcal{O}\left(b^3\right) \tag{68}$$

Notice that this is of second order, even if the truncation is to first order.

Second order ($N = 2$)

Evaluating (59) it becomes

$$1 + b^2F_2 + \mathcal{O}\left(b^3\right) = 1 + b^2\left(F_2 + K_2 + \sin^2\phi - 3\cos^2\phi + 2A_{2,2}\cos 2\phi\right) + \mathcal{O}\left(b^3\right),$$

and using trigonometric relations it becomes

$$1 + b^2F_2 + \mathcal{O}\left(b^3\right) = 1 + b^2(F_2 + K_2 - 1) + 2b^2\left(A_{2,2} - 1\right)\cos 2\phi + \mathcal{O}\left(b^3\right). \tag{69}$$

From this we read off $K_2 = A_{22} = 1$, whereas F_2 is not determined (it will be determined in third order).

Consequently, we have

$$x = -b\sin\phi - b^2\sin 2\phi + \mathcal{O}\left(b^3\right), \qquad y = b\cos\phi + b^2\cos 2\phi + \mathcal{O}\left(b^3\right), \tag{70}$$

and

$$h_0 = \tfrac{1}{2}b^2 + \mathcal{O}\left(b^4\right). \tag{71}$$

Evidently, all odd powers must vanish: $h_{\text{odd}} = 0$.

Ninth order ($N = 9$)

The above procedure can be continued to higher orders. In each order of truncation, N, we only need to determine the constants $A_{N,N}$, $A_{N-2,N}$, ..., until it breaks off. Expressed in terms of the A_n's, the result is to 9'th order (leaving out the explicit $\mathcal{O}\left(b^{10}\right)$),

$$\begin{aligned}
A_1 &= b,\\
A_2 &= b^2 + \tfrac{1}{2}b^4 + \tfrac{29}{12}b^6 + \tfrac{1123}{72}b^8,\\
A_3 &= \tfrac{3}{2}b^3 + \tfrac{19}{12}b^5 + \tfrac{1183}{144}b^7 + \tfrac{475367}{8640}b^9,\\
A_4 &= \tfrac{8}{3}b^4 + \tfrac{313}{72}b^6 + \tfrac{103727}{4320}b^8,\\
A_5 &= \tfrac{125}{24}b^5 + \tfrac{16603}{1440}b^7 + \tfrac{5824751}{86400}b^9,\\
A_6 &= \tfrac{54}{5}b^6 + \tfrac{54473}{1800}b^8,\\
A_7 &= \tfrac{16807}{720}b^7 + \tfrac{23954003}{302400}b^9,\\
A_8 &= \tfrac{16384}{315}b^8,\\
A_9 &= \tfrac{531441}{4480}b^9.
\end{aligned} \tag{72}$$

Stokes himself carried by hand the expansion to 5'th order.

The constants become in the same approximation

$$F = 1 + b^2 + \tfrac{7}{2}b^4 + \tfrac{229}{12}b^6 + \tfrac{6175}{48}b^8, \tag{73}$$

$$K = b^2 + 2b^4 + \tfrac{35}{4}b^6 + \tfrac{1903}{36}b^8, \tag{74}$$

$$h_0 = \tfrac{1}{2}b^2 + b^4 + \tfrac{35}{8}b^6 + \tfrac{1903}{72}b^8. \tag{75}$$

Notice that $K = 2h_0$.

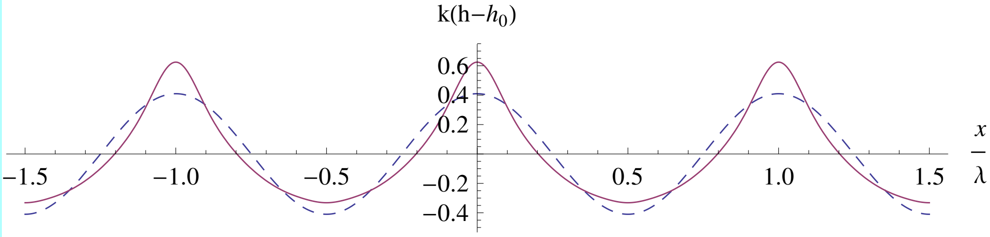


Figure 1. Three periods of the truncated wave for $N = 9$ with $ka = 0.43$ together with the first-order wave (dashed). One notices that the nonlinear effects make the wave crests sharper and taller, and the troughs wider and shallower.

Explicit wave form

For general reasons the wave must take the symmetric form

$$h = h_0 + \sum_{n=1}^{\infty} H_n \cos nx, \tag{76}$$

as a function of x. The coefficients H_n must like the A_n be of the form

$$H_n = \sum_{m=n}^{\infty} H_{nm} b^m \tag{77}$$

with $H_{\text{even,odd}} = H_{\text{odd,even}} = 0$.

These coefficients are obtained successively in the same iterative procedure that led to the A_n, with the result

$$\begin{aligned}
H_1 &= b + \frac{9}{8}b^3 + \frac{769}{192}b^5 + \frac{201457}{9216}b^7 + \frac{325514563}{2211840}b^9,\\
H_2 &= \frac{1}{2}b^2 + \frac{11}{6}b^4 + \frac{463}{48}b^6 + \frac{1259}{20}b^8,\\
H_3 &= \frac{3}{8}b^3 + \frac{315}{128}b^5 + \frac{85563}{5120}b^7 + \frac{5101251}{40960}b^9,\\
H_4 &= \frac{1}{3}b^4 + \frac{577}{180}b^6 + \frac{57703}{2160}b^8,\\
H_5 &= \frac{125}{384}b^5 + \frac{38269}{9216}b^7 + \frac{318219347}{7741440}b^9,\\
H_6 &= \frac{27}{80}b^6 + \frac{30141}{5600}b^8,\\
H_7 &= \frac{16807}{46080}b^7 + \frac{51557203}{7372800}b^9,\\
H_8 &= \frac{128}{315}b^8,\\
H_9 &= \frac{531441}{1146880}b^9.
\end{aligned} \tag{78}$$

Finally we want to express b in terms of the lowest order harmonic amplitude $a = H_1$.

Solving for b we find

$$b = a - \frac{3}{8}a^3 - \frac{5}{24}a^5 - \frac{26713}{9216}a^7 - \frac{25971763}{2211840}a^9, \tag{79}$$

which when inserted into the H's yield

$$\begin{aligned}
H_1 &= a, \\
H_2 &= \frac{1}{2}a^2 + \frac{17}{24}a^4 + \frac{233}{128}a^6 + \frac{348851}{46080}a^8, \\
H_3 &= \frac{3}{8}a^3 + \frac{153}{128}a^5 + \frac{10389}{2560}a^7 + \frac{747697}{40960}a^9, \\
H_4 &= \frac{1}{3}a^4 + \frac{307}{180}a^6 + \frac{31667}{4320}a^8, \\
H_5 &= \frac{125}{384}a^5 + \frac{10697}{4608}a^7 + \frac{47169331}{3870720}a^9, \\
H_6 &= \frac{27}{80}a^6 + \frac{34767}{11200}a^8, \\
H_7 &= \frac{16807}{46080}a^7 + \frac{30380383}{/7372800}a^9, \\
H_8 &= \frac{128}{315}a^8, \\
H_9 &= \frac{531441}{1146880}a^9.
\end{aligned} \tag{80}$$

The 9'th order wave shape is shown in Figure 1 for $a = 0.43$.

References

[1] G. Stokes, *On the Theory of Oscillatory Waves*, Transactions of the Cambridge Philosophical Society VIII (1847) 197–229, and Supplement 314–326.

[2] F. Ursell, *The long-wave paradox in the theory of gravity waves*, Proceedings of the Cambridge Philosophical Society **49** (1953) 685–694.