Averaging method for nonlinear laminar Ekman layers

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We study laminar Ekman boundary layers in rotating systems using an averaging method similar to the technique of von Kármán and Pohlhausen. The method allows us to explore nonlinear corrections to the standard Ekman theory even at large Rossby numbers. We consider both the standard self-similar ansatz for the velocity profile, which assumes that a single length scale describes the boundary layer structure, and a new non self-similar ansatz in which the decay and the oscillations of the boundary layer are described by two different length scales. For both profiles we calculate the up-flow in a vortex core in solid body rotation analytically. We compare the quantitative predictions of the model with von Kármán's exact similarity solution and find that the results for the non self-similar profile are in almost perfect quantitative agreement with the exact solution and performs markedly better than the self-similar profile.

1. Introduction

Ekman layers are boundary layers which form in rotating systems at either free or solid boundaries, typically normal to the axis of rotation. Such boundary layers play important roles in many geophysical and technical flows including large atmospheric vortices and source-sink flows in turbines (Lugt 1995). In the bulk, far from any walls, such flows are essentially two-dimensional with velocities mainly perpendicular to the axis of rotation. Due to the Ekman layer, however, there is also a small but important velocity component along the axis of rotation, which is referred to as either Ekman pumping or suction. A cyclone (larger absolute rotation rate in the bulk than at the boundary) creates Ekman pumping whereas an anti-cyclone (smaller absolute rotation rate in the bulk than at the boundary) creates Ekman suction. The properties of Ekman layers are well-described by linear differential equations when the flow in the rotating reference system is weak compared with the background rotation, i.e., at low Rossby number. The Rossby number is defined as the characteristic value of the ratio of the nonlinear terms and the Coriolis terms in the Navier-Stokes equations in the frame of reference rotating with the background rotation rate (Batchelor 1967). At large values of the Rossby number the nonlinear terms become important, which makes it difficult to solve the governing equations and creates a need for approximate methods.

We model the nonlinear Ekman layers at solid boundaries for flows with rotational symmetry using an averaging method similar to the technique introduced by von Kármán (1921) and Pohlhausen (1921). Owen, Pincombe & Rogers (1985) applied an averaging method to describe laminar and turbulent boundary layers in rotating fluids. They used a self-similar boundary layer profile (described by a single length scale) and focused on a particular geometry in which up-flow from the boundary layer does not occur because of the symmetry of the problem. We extend their equations to include the terms describing up-flow from the boundary layer. The self-similar profile has the same damped oscillatory structure as the solution to the linear Ekman equations but with a boundary layer thickness which depends on the local Rossby number. To investigate the applicability of the method we compare the results with the exact similarity solution due to von Kármán (1921) for a fluid above an infinite rotating disk. The results agree qualitatively, but it turns out that the quantitative agreement is poor, leading, e.g., to a considerable overestimate of the Ekman pumping. We trace this discrepancy to the assumption that a self-similar ansatz for the velocity profile based on a single length scale can describe both the oscillatory and the exponential components of the boundary layer. This is correct for the linear solution but for the nonlinear case there is no basis for such an assumption. We thus introduce a non self-similar profile which involves two length scales and show that the boundary layer structure and the up-flow for the non self-similar profile agree almost perfectly with von Kármán's exact solution at arbitrary (positive) Rossby number.

The layout of the paper is as follows. In § 2 we briefly outline the linear Ekman theory. In § 3 we present the two different averaging methods and derive analytical expressions for the central up-flow velocity in a vortex core in solid body rotation. In § 4 we present numerical solutions of the averaged equations for a generic source-sink vortex. Finally, in § 5 we apply the averaging techniques to the von Kármán flow and compare the results with the exact solution.

2. Linear Ekman theory

We focus on the flow in a container with rotational symmetry which is rotating about its vertical axis of symmetry, and we thus apply the Navier-Stokes equations in polar coordinates written in the frame of reference co-rotating with the container. We let u, v, and w denote the radial, the azimuthal, and the vertical velocity component, respectively. In the bulk of the fluid the flow is approximately two-dimensional, independent of the vertical coordinate z and we denote the azimuthal velocity there by v_0. At the bottom of the container there is a mismatch between the bulk velocity and the rigidly rotating solid bottom, and therefore an Ekman layer is formed (see, e.g., Batchelor 1967). We assume that the vertical velocity component is small and from the vertical Navier-Stokes equation it thus follows that the effective pressure is independent of z and equal to its value in the geostrophic bulk of the fluid, i.e., far above the bottom Ekman layer. At small Rossby number the governing equations reduce to the following linear equations

$$0 = \nu \frac{\partial^2 u}{\partial z^2} + 2\,\Omega\,(v - v_0) \tag{2.1}$$

$$0 = \nu \frac{\partial^2 v}{\partial z^2} - 2\,\Omega\, u\ , \tag{2.2}$$

where Ω is the angular velocity of the container. With the boundary conditions $u(r,z) = v(r,z) = 0$ at $z = 0$, $u(r,z) \to 0$ as $z \to \infty$, and $v(r,z) \to v_0(r)$ as $z \to \infty$, the solution is

$$u(r,z) = -v_0(r)\, e^{-z/\delta} \sin(z/\delta) \tag{2.3}$$

$$v(r,z) = v_0(r)\,[1 - e^{-z/\delta} \cos(z/\delta)]\ , \tag{2.4}$$

where $\delta = \sqrt{\nu/\Omega}$ is the boundary layer thickness. The continuity equation links u and w

$$\frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{\partial w}{\partial z} \; = \; 0 \; , \tag{2.5}$$

and since $w(r,z) = 0$ for $z = 0$ we have

$$w(r,z) \; = \; \frac{\delta}{2\,r}\,\frac{\mathrm{d}(rv_0)}{\mathrm{d}r}\left(\,1 - e^{-z/\delta}\,[\sin(z/\delta) + \cos(z/\delta)]\,\right) \, . \tag{2.6}$$

It follows that the vertical velocity component in the bulk of the fluid is proportional to the z-component of the vorticity there

$$w_0 \; = \; \frac{\delta}{2\,r}\,\frac{\mathrm{d}(rv_0)}{\mathrm{d}r} \; . \tag{2.7}$$

Depending on the sign of w_0, this is referred to as Ekman pumping and Ekman suction. If the fluid is rotating as a solid body with $v_0 \; = \; C\,r$ it follows that

$$w_0 \; = \; \left(\frac{\nu}{\Omega}\right)^{1/2} \, C \, . \tag{2.8}$$

A cyclonic vortex of this kind thus generates up-flow whereas and anti-cyclonic vortex gives rise to down-flow. In the following we calculate nonlinear corrections to (2.8) using the averaged equations.

3. Averaged boundary layer equations

3.1. *Governing equations*

The linear equations (2.1) and (2.2) are only valid when the flow in the co-rotating reference system is weak and the nonlinear terms can be neglected compared to the Coriolis terms, i.e., at small Rossby number. At larger values of the Rossby number we approximate the radial and the azimuthal Navier-Stokes equations by boundary layer equations in which we neglect the derivatives with respect to r in the viscous terms

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2 - v_0^2}{r} = \nu\,\frac{\partial^2 u}{\partial z^2} + 2\Omega(v - v_0) \tag{3.1}$$

$$u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \nu\,\frac{\partial^2 v}{\partial z^2} - 2\Omega u \; , \tag{3.2}$$

where $v(r,z) \to v_0(r)$ when $z \to \infty$. Using the continuity equation (2.5) we rewrite the Navier-Stokes equations in a form which is easy to integrate with respect to z

$$\frac{1}{r}\frac{\partial(ru^2)}{\partial r} + \frac{\partial(uw)}{\partial z} - \frac{v^2 - v_0^2}{r} = \nu\,\frac{\partial^2 u}{\partial z^2} + 2\Omega(v - v_0) \tag{3.3}$$

$$\frac{1}{r^2}\frac{\partial(r^2uv)}{\partial r} + \frac{\partial(vw)}{\partial z} = \nu\,\frac{\partial^2 v}{\partial z^2} - 2\Omega u \; . \tag{3.4}$$

These two equations are used in the following as the starting point in the derivation of the averaged equations.

3.2. *Self-similar profile*

In the averaging method we make an ansatz on the z-dependence of the velocity components, i.e., on the boundary layer structure. Following Owen, Pincombe & Rogers (1985) we make a straight forward generalization of (2.3) – (2.4) and consider the self-similar profile

$$u(r,z) = u_0\, f(z/\delta) \tag{3.5}$$
$$v(r,z) = v_0\, [1 - g(z/\delta)] \ , \tag{3.6}$$

where u_0, v_0, and δ are functions of r, and where

$$f(\eta) \equiv \exp(-\eta) \sin\eta \tag{3.7}$$
$$g(\eta) \equiv \exp(-\eta) \cos\eta \ . \tag{3.8}$$

The term "self-similar" is used here to emphasize that the profile does not depend on both r and z but only on the combination $z/\delta(r)$. The governing equations (3.3) and (3.4) are integrated with respect to z from zero to infinity, and the resulting averaged equations read

$$I_2 \frac{1}{r}\frac{\mathrm{d}(r u_0^2 \delta)}{\mathrm{d}r} + (2I_4 - I_5)\frac{v_0^2}{r}\delta = -\frac{\nu u_0}{\delta} f'(0) - 2 I_4 \Omega v_0 \delta \tag{3.9}$$

$$(I_1 - I_3)\frac{1}{r^2}\frac{\mathrm{d}(r^2 u_0 v_0 \delta)}{\mathrm{d}r} - I_1 \frac{v_0}{r}\frac{\mathrm{d}(r u_0 \delta)}{\mathrm{d}r} = \frac{\nu v_0}{\delta} g'(0) - 2 I_1 \Omega u_0 \delta \ , \tag{3.10}$$

where the following definitions of integrals are used

$$I_1 = \int_0^\infty \mathrm{d}\eta f(\eta) = \frac{1}{2}\,, \qquad I_2 = \int_0^\infty \mathrm{d}\eta f^2(\eta) = \frac{1}{8} \tag{3.11}$$

$$I_3 = \int_0^\infty \mathrm{d}\eta f(\eta) g(\eta) = \frac{1}{8}\,, \qquad I_4 = \int_0^\infty \mathrm{d}\eta g(\eta) = \frac{1}{2} \tag{3.12}$$

$$I_5 = \int_0^\infty \mathrm{d}\eta g^2(\eta) = \frac{3}{8}\ . \tag{3.13}$$

Equations (3.9) and (3.10) are written for a general self-similar profile as defined by (3.5) and (3.6). The integrals can be evaluated as shown in (3.11) – (3.13) for the ansatz (3.7) and (3.8), and the governing equations become

$$\frac{1}{8r}\frac{\mathrm{d}(r u_0^2 \delta)}{\mathrm{d}r} + \frac{5 v_0^2}{8r}\delta = -\frac{\nu u_0}{\delta} - \Omega v_0 \delta \tag{3.14}$$

$$\frac{3}{8r^2}\frac{\mathrm{d}(r^2 u_0 v_0 \delta)}{\mathrm{d}r} - \frac{v_0}{2r}\frac{\mathrm{d}(r u_0 \delta)}{\mathrm{d}r} = -\frac{\nu v_0}{\delta} - \Omega u_0 \delta \ . \tag{3.15}$$

Notice that w does not appear in the averaged equations. The flow rate, q, of the radial inflow is

$$q \equiv -2\pi r \int_0^\infty \mathrm{d}z\ u = -2\pi r u_0 \int_0^\infty \mathrm{d}z f(z/\delta) = -\pi r u_0 \delta \ . \tag{3.16}$$

Owen, Pincombe & Rogers (1985) assumed that q is independent of r and therefore the second term on the left hand side of (3.15) does not appear in their averaged azimuthal equation.

With linear velocity profiles $u_0 = -A\,r$ and $v_0 = C\,r$, we find that δ becomes independent of r and the averaged equations (3.14) and (3.15) reduce to the algebraic equations

$$\frac{3}{8}A^2 + \frac{5}{8}C^2 = \frac{\nu\,A}{\delta^2} - \Omega\,C \tag{3.17}$$

$$-\frac{1}{2}A\,C = -\frac{\nu\,C}{\delta^2} + \Omega\,A\,, \tag{3.18}$$

In this special case the Rossby number is by definition

$$\mathsf{Ro} \equiv \frac{C}{2\,\Omega}\,. \tag{3.19}$$

The vertical velocity component in the bulk of the fluid, w_0, is through the continuity equation (2.5) related to the radial velocity component u_0

$$w_0 = -\frac{1}{2\,r}\frac{\mathrm{d}(ru_0\delta)}{\mathrm{d}r}\,, \tag{3.20}$$

and thus it follows that the assumption of a linear radial velocity profile leads to

$$w_0 = \delta\,A\,. \tag{3.21}$$

By solving (3.17) and (3.18) for δ and A it follows that the nonlinear Ekman pumping velocity is reduced in comparison with the linear result as the Rossby number is increased

$$\frac{w_0}{(\nu/\Omega)^{1/2}\,C} = \left[\frac{4 + 5\,\mathsf{Ro}}{(4+\mathsf{Ro})(1+\mathsf{Ro})^2}\right]^{1/4}\,. \tag{3.22}$$

At $\mathsf{Ro} = 0$ the right hand side is equal to one in agreement with linear Ekman theory (2.8), and in the limit of infinite Rossby number w_0 is independent of Ω and we have $w_0 = 20^{1/4}\sqrt{\nu\,C} \approx 2.115\,\sqrt{\nu\,C}$.

3.3. *Non self-similar profile*

As we will show in § 5 the self-similar profile leads to predictions which are not in quantitative agreement with von Kármán's exact similarity solution. We thus introduce the following non self-similar boundary layer profile †

$$u(r,z) = -v_0\,e^{-z/\delta_1}\sin(z/\delta_2) \tag{3.23}$$

$$v(r,z) = v_0\,[\,1 - e^{-z/\delta_1}\cos(z/\delta_2)\,]\,, \tag{3.24}$$

where v_0, δ_1, and δ_2 are functions of r. Like the self-similar profile this is a generalization of (2.3) – (2.4) but now with two length scales describing respectively the decay and

† In this expression we have not allowed for different amplitudes on the radial and azimuthal fields. Allowing this requires another condition to close the system of equations. We have investigated the case where this other condition is the standard wall curvature compatibility condition, $\nu\,\frac{\partial^2 u}{\partial z^2}\Big|_{z=0} = 2\Omega v_0 + \frac{v_0^2}{r}$. The resulting equations are considerably more complex, and lead in the end to no appreciable improvement in the problems that we have considered.

the oscillations in the boundary layer structure. At low Rossby number we have $\delta_1 = \delta_2 = \sqrt{\nu/\Omega}$ but in general the two length scales are different. The presence of two length scales in the solution is apparent in the exact asymptotic solution at large z as shown by Bödewadt (1940) and Rogers & Lance (1960). Our ansatz has the same structure as the special asymptotic solution which also satisfies the no-slip boundary condition at $z = 0$. The averaged equations are in this case

$$\frac{1}{r}\frac{\mathrm{d}(r v_0^2 K_2)}{\mathrm{d}r} + \frac{v_0^2}{r}(2K_4 - K_5) = \frac{\nu v_0}{\delta_2} - 2\Omega v_0 K_4 \tag{3.25}$$

$$\frac{1}{r^2}\frac{\mathrm{d}(r^2 v_0^2 K_3)}{\mathrm{d}r} - \frac{v_0 K_1}{r}\frac{\mathrm{d}(r v_0)}{\mathrm{d}r} = -\frac{\nu v_0}{\delta_1} + 2\Omega v_0 K_1\,, \tag{3.26}$$

where we define

$$K_1 = 4K_3 = \frac{\delta_1^2\,\delta_2}{\delta_1^2 + \delta_2^2}\,, \qquad K_2 = \frac{\delta_1^3}{4(\delta_1^2 + \delta_2^2)} \tag{3.27}$$

$$K_4 = \frac{\delta_1\,\delta_2^2}{\delta_1^2 + \delta_2^2}\,, \qquad K_5 = K_2 + \frac{1}{2}K_4\,. \tag{3.28}$$

With the linear velocity profile $v_0 = C\,r$, the averaged equations reduce to

$$C\left(\frac{1}{2}\delta_1^2 + \frac{3}{2}\delta_2^2\right) = \nu\,\frac{\delta_1^2 + \delta_2^2}{\delta_1\delta_2} - 2\,\Omega\,\delta_2^2 \tag{3.29}$$

$$-C\,\delta_1^2 = -\nu\,\frac{\delta_1^2 + \delta_2^2}{\delta_1\delta_2} + 2\,\Omega\,\delta_1^2\,, \tag{3.30}$$

under the assumption that δ_1 and δ_2 do not depend on r. Using the non self-similar profile we thus predict that the nonlinear Ekman pumping velocity is reduced in the following way in comparison with the linear result

$$\frac{w_0}{(\nu/\Omega)^{1/2}\,C} = \frac{((2+\mathsf{Ro})(2+3\mathsf{Ro}))^{1/4}}{\sqrt{2}(1+\mathsf{Ro})} \tag{3.31}$$

Similar to (3.22) the right hand side is equal to one when $\mathsf{Ro} = 0$, and in the limit of infinite Ro we have $w_0 = 3^{1/4}\sqrt{\nu\,C} \approx 1.316\sqrt{\nu\,C}$. The non self-similar profile thus gives a weaker up-flow than the self-similar profile in the limit of large Rossby number.

4. Numerical solution for source-sink vortex

Source-sink flows in a rotating cylindrical container are important examples of vortex flows controlled by Ekman layers. In this paragraph we consider a generic source-sink vortex and assume an azimuthal velocity profile of the form

$$v_0 = \frac{\Gamma}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{\kappa^2}\right)\right]\,. \tag{4.1}$$

The velocity profile is linear when $r \ll \kappa$, and it is like a line vortex with circulation Γ when $r \gg \kappa$. This is similar to a Rankine vortex, and the profile is thus like a smooth Rankine vortex with a soft transition between the vortex core and the outside velocity field. It follows by definition that the z-component of vorticity has a Gaussian profile and in linear Ekman theory we thus predict that the up-flow is

$$w_0 \;=\; \frac{\sqrt{\nu/\Omega}\,\Gamma}{2\pi\kappa^2}\,\exp\left(-\frac{r^2}{\kappa^2}\right) \;. \tag{4.2}$$

The vertical velocity component in the bulk of the fluid, w_0, is through the continuity equation related to q

$$w_0 \;=\; \frac{1}{2\pi r}\frac{\mathrm{d}q}{\mathrm{d}r} \;, \tag{4.3}$$

and integration gives the following expression for q in terms of w_0

$$q(r) \;=\; Q - 2\pi\int_r^\infty \mathrm{d}x w_0(x)x \;, \tag{4.4}$$

where Q is defined as the limit value of q for large r. At $r \gg \kappa$ linear Ekman theory is valid since the local Rossby number there is small, and the circulation Γ is thus linked to the flow-rate Q. The velocity profiles now become

$$v_0 = \frac{Q}{\pi\,\sqrt{\nu/\Omega}\,r}\left[1 - \exp\left(-\frac{r^2}{\kappa^2}\right)\right] \tag{4.5}$$

$$w_0 = \frac{Q}{\pi\kappa^2}\,\exp\left(-\frac{r^2}{\kappa^2}\right) \;. \tag{4.6}$$

To obtain numerical solutions for the self-similar profile we eliminate the boundary layer thickness δ in the averaged equations (3.9) and (3.10) and use u_0, v_0, and q as the dependent variables. We write the equations in the following form assuming the azimuthal velocity component v_0 to be known

$$u_0' = -\frac{3u_0v_0'}{v_0} - \frac{3u_0}{r} - \frac{5v_0^2}{ru_0} - \frac{16\pi^2\nu r^2u_0^2}{q^2} - 8\Omega\left(\frac{u_0}{v_0} + \frac{v_0}{u_0}\right) \tag{4.7}$$

$$q' = \frac{3qv_0'}{v_0} + \frac{3q}{r} + \frac{8\pi^2\nu r^2u_0}{q} + \frac{8\Omega q}{v_0} \;, \tag{4.8}$$

and supplement the equations by boundary conditions at large $r = R$:

$$u_0(R) = -\frac{Q}{\pi\,\sqrt{\nu/\Omega}\,R} \tag{4.9}$$

$$q(R) = Q \;. \tag{4.10}$$

To solve the equations for the non self-similar profile numerically we use the functions $\alpha \equiv K_1$ and $\beta \equiv \delta_1/\delta_2$ and rewrite equations (3.25) and (3.26) in terms of them as

$$\alpha' = \frac{2}{v_0}\left[\left(v_0' + \frac{v_0}{r} + 4\Omega\right)\alpha - \frac{2\nu}{\delta_1}\right] \tag{4.11}$$

$$\beta' = -\frac{1}{2\alpha v_0}\left[\left(5v_0' + \frac{4v_0}{r} + 16\Omega\right)\alpha\beta + \left(\frac{3v_0}{r} + 4\Omega\right)\frac{\alpha}{\beta} - \frac{10\nu}{\delta_2}\right] \;. \tag{4.12}$$

We supplement the equations by the following boundary conditions at large $r = R$:

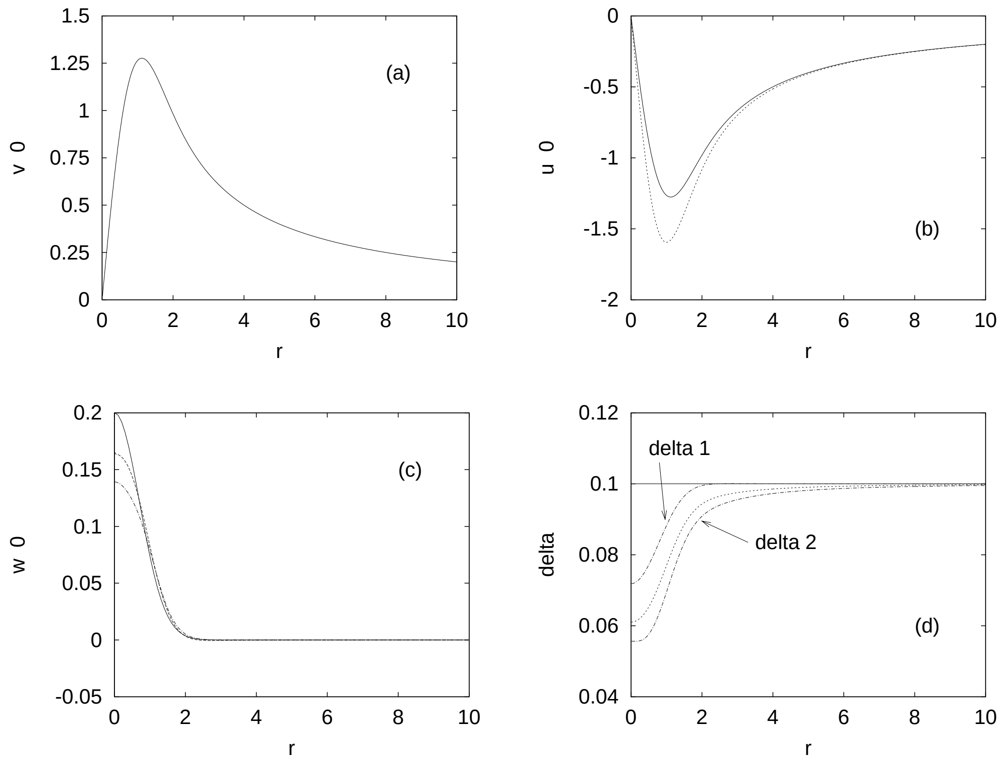

FIGURE 1. Numerical solution of the averaged equations for the source-sink vortex. (a) The assumed azimuthal velocity v_0, (b) the radial velocity u_0, (c) the vertical velocity w_0, and (d) the boundary layer thickness δ all computed at a central Rossby number of 1. The solid curves show the linear theory, the dashed curves the result obtained with the self-similar profile, and the dot-dashed curves the result obtained with the non self-similar profile. In (b) the solid and the dot-dashed curves are identical by assumption and thus only the solid curve is shown.

$$\alpha(R) = \frac{1}{2}\sqrt{\frac{\nu}{\Omega}} \tag{4.13}$$

$$\beta(R) = 1 \ . \tag{4.14}$$

Figure 1 shows the solution for u_0, w_0, and δ with $\nu = 0.01$, $\Omega = 1$, $\kappa = 1$, and $Q = \pi/5$, which corresponds to a central Rossby number of 1. At large values of r the solution is well-described by the linear theory, whereas the boundary layer thickness decreases closer to the vortex center compared to the constant linear Ekman layer thickness. The central up-flow velocities are described quantitatively by (3.22) and (3.31) with $\mathsf{Ro} = 1$.

5. The similarity solution by von Kármán

In this paragraph we investigate the applicability of the self-similar and the non self-similar boundary layer profile by comparing them with the exact similarity solution introduced by von Kármán. The flow above a flat rotating disk of infinite extension was studied by von Kármán (1921) who introduced the following ansatz for the velocity field

$$u(r,z) = r\,F(z) \tag{5.1}$$
$$v(r,z) = r\,G(z) \tag{5.2}$$
$$w(r,z) = H(z)\,. \tag{5.3}$$

With this ansatz the Navier-Stokes equations reduce to a set of ordinary differential equations for F, G, and H. The Navier-Stokes equations and the continuity equation in the frame of reference rotating with angular velocity Ω become

$$F^2 + H\,F' - G^2 + C^2 = \nu\,F'' + 2\,\Omega\,(G - C) \tag{5.4}$$
$$2\,F\,G + H\,G' = \nu\,G'' - 2\,\Omega\,F \tag{5.5}$$
$$H\,H' = -P' + \nu\,H'' \tag{5.6}$$
$$2\,F + H' = 0\,, \tag{5.7}$$

which are to be supplemented by the five boundary conditions

$$F(0) = 0 \qquad F(\infty) = 0 \tag{5.8}$$
$$G(0) = 0 \qquad G(\infty) = C \tag{5.9}$$
$$H(0) = 0 \tag{5.10}$$

In the original problem considered by von Kármán the azimuthal velocity goes to zero far above the disk in laboratory frame corresponding to $C = -\Omega$. Similarly the problem considered by Bödewadt (1940) of a fluid in solid body rotation above a fixed disk is obtained with $\Omega = 0$. Linear Ekman theory (2.3), (2.4), and (2.6) is valid at small values of C.

Figure 2 shows numerical solutions (symbols) of von Kármán's equations (5.4) – (5.7) together with results of the averaging method with the self-similar profile (dashed curves) and the non self-similar profile (solid curves). The solution with the non self-similar profile has $\delta_1 \geq \delta_2$ and it captures the damped oscillatory structure of the exact solution at $\mathsf{Ro} = 4$ markedly better than the self-similar profile. This is particularly evident for the vertical velocity component shown in figure 2(d) for Rossby numbers between 0 and 8. The asymptotic behavior of the exact solution in the limit of infinite Rossby number was described by Bödewadt (1940) who found that $w_0 = 1.349\sqrt{\nu\,C}$. The prediction, $w_0 \approx 1.316\sqrt{\nu\,C}$, obtained using the non self-similar profile agrees well with this result whereas the method using the self-similar profile overestimates the up-flow velocity, $w_0 \approx 2.115\sqrt{\nu\,C}$.

6. Conclusions

We have analyzed two methods based on averaging for computing the structure of nonlinear Ekman layers. We find that a non self-similar velocity profile with separate length scales for the decay and the oscillations, respectively, give very accurate predictions. We believe that this method will be useful for many important flow configurations and hope that it will be confronted with accurate measurements of the up-flow in well-controlled laboratory experiments and ideally with velocity measurements resolving the Ekman layer structure.

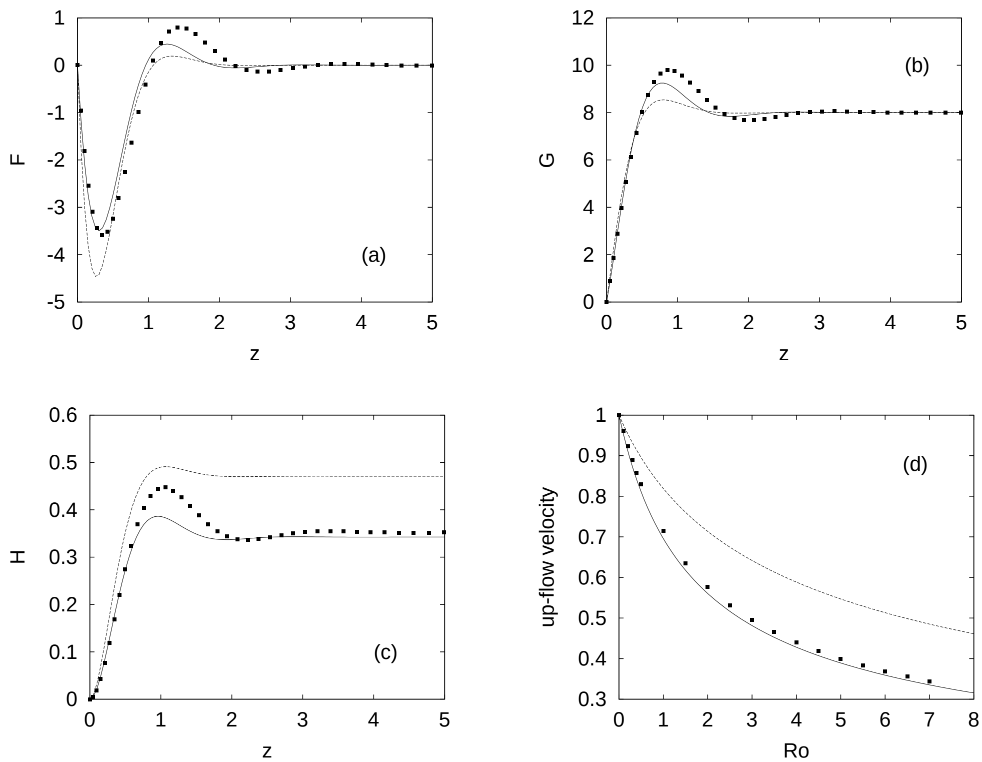

FIGURE 2. Numerical solutions of the averaged equations for the von Kármán flow at $\mathsf{Ro} = 4$. (a) – (c) the boundary layer structure of u, v, and w, and (d) the ratio between the nonlinear up-flow velocity and the result from linear Ekman theory as function of Ro. The symbols show the numerical solution of (5.4) – (5.7), the dashed curves the result obtained with the self-similar profile, and the solid curves the result obtained with the non self-similar profile.

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